Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes

Lectures on Hilbert schemes

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Contents

1. Introduction

These are slightly extended notes of my lectures on the geometry of Hilbert schemes of points that I gave at the workshop on Algebraic structures and moduli spaces at CRM Montreal in July 2003.

The purpose of the lectures was to provide an introduction to some aspects of the theory of Hilbert schemes of points on surfaces. It seems to me that the interest in Hilbert schemes has many sources: There are relations to enumerative geometry, to moduli spaces of sheaves, to the combinatorics of the symmetric group, to orbifold cohomology, to the generalised McKay correspondence, and they provide almost all known examples of irreducible symplectic manifolds (up to deformation).

The scope of these lectures is much narrower. I explain the classical theorems of Fogarty and Briançon based on a technique due to Ellingsrud and Strømme and give a fairly self-contained presentation of Nakajima's theorem on the Heisenberg action on the cohomology of Hilbert schemes with full proofs. I then explain work of Li, Qin and Wang, and of Sorger and myself on the ring structure of the cohomology of Hilbert schemes. In this part proofs are only sketched, or some of the ingredients are explained. The notes end with a brief discussion on the relation to a conjecture of Ruan. Besides the orginal references, it is worth while also to look at other review articles or lecture notes with the same or a similar subject by Nakajima [**33**], Göttsche [**15**], Göttsche and Ellingsrud [**5**], Wang [**38**], Qin and Wang [**34**]. What is perhaps more interesting is what is not covered here: the relation to the theory of symplectic varieties (cf. the notes [**25**]); the relation to moduli of sheaves [**31**]; the work of Haiman on the $n!$-conjecture and related questions [**22, 23**]; Nakamura's G-Hilbert schemes and the McKay correspondence [**2**]. The reader is invited to look up detailed bibliographies in the quoted articles.

2. Configurations of unordered n-tuples

Configuration spaces of n-tuples of points that move on a given manifold X are interesting topological spaces with fascinating topological and geometric properties. There is a basic difference in whether one is interested in ordered or unordered n-tuples. As a first approach to a proper definition of the right configuration space, we may consider the product X^n for ordered tuples and its quotient $S^n(X) := X^n/S_n$ by the symmetric group for unordered tuples. In both spaces, points that correspond to n-tuples of points that are not pairwise distinct are special: In X^n they have non-trivial isotropy groups with respect to the S_n-action, in S^nX they are singular (except for the special case $\dim(X) = 1$). One is therefore interested in other smooth compactifications of the configuration space of n-tuples of distinct points. For ordered n-tuples a very nice compactification was constructed by Fulton and MacPherson [**13**]. For unordered n-tuples a smooth natural compactification for all n exists so far only for curves, where the symmetric product is already smooth, and for surfaces, where such a compactification is provided by the so-called Hilbert schemes of points.

Assume that X is a smooth quasiprojective scheme of finite type over $\mathbb{C}$. Let $Z \subset X$ be a zero-dimensional subscheme. Then $H^0(Z, \mathcal{O}_Z)$ is an artinian $\mathbb{C}$-algebra. By definition, the length of Z is the length of this algebra, $\ell(Z) := \operatorname{length} H^0(\mathcal{O}_Z) = \dim_{\mathbb{C}} H^0(\mathcal{O}_Z)$. Let $\mathrm{Hilb}^n(X)$ denote the set of all zero-dimensional subschemes $Z \subset X$ of length n. We will see in the next section how $\mathrm{Hilb}^n(X)$ can be given a natural scheme structure. Assume now that X is reduced. If $x \in Z$ is a closed point, the multiplicity of x in Z is defined as $\dim_{\mathbb{C}}(\mathcal{O}_{Z,x})$. We may associate to Z the cycle $|Z|$ corresponding to the underlying set counted with multiplicities. $|Z|$

is a point in the symmetric product:

$$|Z| := \sum_{x\in X} \dim_{\mathbb{C}}(\mathcal{O}_{Z,x})\cdot x \ \in \ S^n(X).$$

Sending Z to $|Z|$ defines the so-called Hilbert-Chow map

$$\rho : \mathrm{Hilb}^n(X) \to S^n X.$$

If X is a smooth curve, ρ turns out to be an isomorphism for all n. Essentially, it suffices to understand the case of the affine line: the symmetric group acts on the coordinate ring $\mathbb{C}[x_1,\ldots,x_n]$ of $(\mathbb{C}^1)^n$ by permuting the variables. The invariant ring is generated by the elementary symmetric functions $\sigma_1 = x_1+\ldots+x_n, \ldots, \sigma_k = \sum_{i_1<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k}, \ldots, \sigma_n = x_1x_2\cdots x_n$. These are algebraically independent. It follows that

$$S^n(\mathbb{C}) = \mathrm{Spec}(\mathbb{C}[x_1,\ldots,x_n]^{S_n}) = \mathrm{Spec}(\mathbb{C}[\sigma_1,\ldots,\sigma_n])) = \mathbb{C}^n.$$

On the other hand, any zero-dimensional subscheme in $\mathbb{C}$ is described by a single polynomial $f(T) = T^n + a_1T^{n-1} + \ldots + a_n$ of degree n. The coefficients a_i define a point $a \in \mathbb{C}^n$, and conversely, any such point defines a polynomial and thus a subscheme. If Z consists of n distinct points $p_1,\ldots,p_n \in \mathbb{C}$, the coefficients of f are precisely the elementary symmetric functions in the p_i: $a_i = \sigma_i(p_1,\ldots,p_n)$.
For arbitrary X, the Hilbert-Chow map ρ is still an isomorphism over the open subset of S^nX corresponding to n-tuples of distinct points in X. The opposite of a tuple of distinct points is a subscheme Z with multiplicity n in a single point $x \in X$: consider the fibre $B^n_x(X) := \rho^{-1}(n\cdot x)$. A point in $B^n_x(X)$ corresponds to a subscheme $Z \subset X$ with underlying set $\{x\}$ and length n. Any such subscheme is given by an ideal $I \subset \mathcal{O}_{X,x}$ with $\dim_{\mathbb{C}} \mathcal{O}_{X,x}/I = n$. Let $\mathfrak{m}$ denote the maximal ideal in $\mathcal{O}_{X,x}$. Since $\dim_{\mathbb{C}} \mathcal{O}_{X,x}/I = n$, one must have $\mathfrak{m}^n \subset I$. Thus I is completely determined by the ideal $\bar{I} := I/\mathfrak{m}^n \subset \mathcal{O}_{X,x}/\mathfrak{m}^n$. We conclude that if X is smooth of dimension d, then $B^n_x(X)$ does not depend on either X or x, and that in order to study $B^n_x(X)$ we may as well assume that $X = \mathbb{C}^d$ and that x is the origin O.
Here is a description of B^n_O for the case of surfaces, i.e. $d = 2$ and small n: B^n_O contains all ideals $I \subset \mathbb{C}[x,y]$ with $\dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n$ and $\mathfrak{m}^n \subset I \subset \mathfrak{m} := (x,y)$.
Case $n = 1$. The only choice is $I = \mathfrak{m}$ and B^1_O is a point, as one expects.
Case $n = 2$. Ideals I correspond to one-dimensional linear subspaces $I/\mathfrak{m}^2 \subset \mathfrak{m}/\mathfrak{m}^2 = \mathbb{C}\langle\bar{x},\bar{y}\rangle$. Thus $B^2_O \cong \mathbb{P}^1 = \mathbb{P}(T_O\mathbb{C}^2)$. This can be geometrically interpreted as follows: if (p_1,p_2) is a pair of distinct points in $\mathbb{C}^2$ that moves towards the origin and collides there then the limiting point in $S^2(\mathbb{C}^2)$ only remembers the number of points that collided and the position where the accident happened whereas the

limiting in $\mathrm{Hilb}^n(\mathbb{C}^2)$ also remembers the limiting direction of the line that passes through p_1 and p_2.

Case $n = 3$**.** Here something interesting happens: there are two qualitatively different types of ideals. One is easy to find: $I_\infty = \mathfrak{m}^2$. It is distinguished from the others by the fact the tangent space $T_O Z_\infty$ of the corresponding subscheme Z_∞ has dimension 2. The other subschemes are contained in the germ of a smooth curve and are therefore called curvilinear. Their ideals have the form $I = (y+\alpha x+\beta x^2, x^3)$ (or x and y exchanged). All these ideals are parametrised by a line bundle over $\mathbb{P}^1$. The subscheme Z_∞ arises as the limiting point of a sequence of triples that approach the origin from three different directions, whereas points of the second type arise as limiting points of triples that move towards the origin along a smooth curve. The affine bundle is compactified by adding the point Z_∞ at infinity. In fact, one can write down an explicit model for B^3_O and a family of schemes parameterised by B^3_O: consider in $\mathbb{P}^4 = \mathrm{Proj}(\mathbb{C}[a, b, c, d, e])$ the projective cone over a rational normal cubic given by the equations $ac - b^2$, $ad - bc$,$bd - c^2$. The family $\mathcal{Z} \subset B^3_O \times \mathbb{C}^2$ is the zero set of the ideal $(ax + by + ex^2, bx + cy + exy, cx + dy + ey^2)$.

For higher n, the general picture is this: call $Z \in B^n_O$ curvilinear, if Z is contained in a germ of a smooth curve, or equivalently, if $\dim_{\mathbb{C}} T_O Z = 1$. Choosing local coordinates in such a way that the line $\{y = 0\}$ is the tangent line to Z at the origin O, we may describe Z by an ideal $I = (y + \alpha_2 x^2 + \ldots + \alpha_{n-1}x^{n-1}, x^n$. The set $B^{n,\mathrm{curv}}_O$ of such I is a bundle over $\mathbb{P}^1$ with fibres $\cong \mathbb{C}^{n-1}$. It is, however, not a vector bundle, since the transition functions are not linear. In any case, $B^{n,\mathrm{curv}}_O$ is irreducible of dimension $n - 1$. By a result of Briançon [**1**], $B^{n,\mathrm{curv}}_O$ is dense in B^n_O.

If $n \geq 2$ and $\dim(X) \geq 2$, the symmetric product $S^n X$ is singular along the locus of tuples that correspond to non-distinct points. On the other hand there is the following miraculous fact:

Theorem 2.1. *(Fogarty [**10**]) — If X is a smooth surface then for all $n \in \mathbb{N}_0$, the Hilbert scheme $\mathrm{Hilb}^n(X)$ is smooth. In particular, $\rho : \mathrm{Hilb}^n(X) \to S^n(X)$ is a resolution of the singularities of the symmetric product.*

This fact is at the base of all the interesting features of the Hilbert scheme of surfaces. For higher dimensional varieties the Hilbert scheme is not only not smooth but in general even more singular than the symmetric product. We will prove Fogarty's theorem in the next section.

3. The Geometry of Hilbert schemes

3.1. Families of subschemes and the moduli functor. Let X denote a quasiprojective scheme over $\mathbb{C}$. By definition, a flat family of proper subschemes in

X parametrised by S is a closed subscheme $Z \subset S \times X$ such that the projection $Z \to S$ is flat and proper. If $s \in S$ is a closed point, we denote the fibre of Z over s by Z_s. Given such a flat family and a morphism $f : S' \to S$, the family $Z' := (f \times \mathrm{id}_X)^{-1}(Z) \subset S' \times X$ is again flat and proper over S'. In this way, we obtain a functor

$$\underline{\mathrm{Hilb}}(X) : (\mathrm{Schemes})^{\mathrm{op}} \longrightarrow (\mathrm{Sets}) \tag{1}$$

that associates to S the set of all flat families of proper subschemes in X that are parameterised by S. Let H be an ample Cartier divisor on X. For any proper subscheme $Z \subset X$ the Hilbert polynomial of Z is defined by $P_Z(n) := \chi(\mathcal{O}_Z \otimes \mathcal{O}_X(nH))$. For a flat family $Z \subset S \times X$ the function $S \ni s \mapsto P_{Z_s} \in \mathbb{Q}[T]$ is locally constant. This implies that for a given polynomial P, the functor

$$\underline{\mathrm{Hilb}}^P(X) : S \mapsto \{Z \subset S \times X | Z \text{ proper and flat}/S, P(Z_s) = P \text{ for all } s \in S\} \tag{2}$$

is an open and closed subfunctor of $\underline{\mathrm{Hilb}}(X)$. The dimension of a subscheme can be read off from the degree of its Hilbert polynomial. The functor $\underline{\mathrm{Hilb}}^n(X)$ associated to the constant polynomial $P = n$ parameterises all zero-dimensional subschemes of length n.

THEOREM 3.1. *(Grothendieck* **[18]***) — The functor $\underline{\mathrm{Hilb}}^P(X)$ is represented by a quasiprojective scheme $\mathrm{Hilb}^P(X)$. If X is projective then $\mathrm{Hilb}^P(X)$ is also projective.*

The theorem then implies that there exists a universal subscheme

$$\Xi_P \subset \mathrm{Hilb}^P(X) \times X, \tag{3}$$

flat over $\mathrm{Hilb}^P(X)$, such that for any $Z \in \underline{\mathrm{Hilb}}^P(X)(S)$ there is a unique classifying morphism $f : S \to \mathrm{Hilb}^P(X)$ such that $Z \cong (f \times \mathrm{id}_X)^*(\Xi_P)$.
In the following we will only need the case $P = n = $ const., and the word Hilbert scheme will always refer to Hilbert schemes of points. Note that $\mathrm{Hilb}^n(X)$ does not depend on the choice of the ample line bundle H.
Whenever a scheme or variety represents a functor one gets an intrinsic description of the Zariski tangent space at closed points. For the Hilbert scheme this takes the following form:

THEOREM 3.2. *(Grothendieck) — Let $[Z] \in \mathrm{Hilb}^n(X)$ be a closed point representing a subscheme $Z \subset X$ with ideal sheaf I_Z and structure sheaf $\mathcal{O}_Z$. Then there is a canonical isomorphism*

$$T_{[Z]} \mathrm{Hilb}^n(X) \cong \mathrm{Hom}_{\mathcal{O}_X}(I_Z, \mathcal{O}_Z). \tag{4}$$

Proof. Let $\mathbb{C}[\varepsilon] = \mathbb{C}[t]/(t^2)$ denote the ring of dual numbers. A tangent vector at $[Z] : \operatorname{Spec} \mathbb{C} \to \operatorname{Hilb}^n(X)$ corresponds to a morphism $\tau : \operatorname{Spec} \mathbb{C}[\varepsilon] \to \operatorname{Hilb}^n(X)$ that restricts to $[Z]$ at the closed point. In the modular interpretation τ corresponds to an ideal $\tilde{I} \subset \mathcal{O}_X[\varepsilon]$ such that the quotient sheaf $\tilde{\mathcal{O}} = \mathcal{O}_X[\varepsilon]/\tilde{I}$ is flat over $\mathbb{C}[\varepsilon]$ and restricts to $\mathcal{O}_Z$ at the closed point of $\operatorname{Spec} \mathbb{C}[\varepsilon]$. We obtain a commutative diagram with exact columns and rows:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & I_Z \arrow[r] \arrow[u] & \mathcal{O}_X \arrow[r] \arrow[u] & \mathcal{O}_Z \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \tilde{I} \arrow[r] \arrow[u] & \mathcal{O}_X[\varepsilon] \arrow[r] \arrow[u] & \tilde{\mathcal{O}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I_Z \arrow[r] \arrow[u] & \mathcal{O}_X \arrow[r] \arrow[u, "\cdot\varepsilon"'] & \mathcal{O}_Z \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{5}$$

That the last row of the diagram is isomorphic to the first row is equivalent to the flatness of $\tilde{\mathcal{O}}$ and $\tilde{I}$. Thus giving a tangent vector amounts to finding an ideal $\tilde{I}$ that fits into the diagram above. As $\tilde{I}$ always contains εI_Z, it is completely determined by the embedding of the quotient $\tilde{I}/I_Z \cong I_Z$ into $\mathcal{O}_X[\varepsilon]/\varepsilon I_Z = \mathcal{O}_X \oplus \varepsilon\mathcal{O}_Z$. Of the two components of this inclusion, the first is the inclusion $I_Z \to \mathcal{O}_X$, the second a homomorphism $t : I_Z \to \mathcal{O}_Z$. In this way, $\tilde{I}$ determines an element $t \in \operatorname{Hom}(I_Z, \mathcal{O}_Z)$, and, conversely, any such t defines an ideal $\tilde{I}$ and hence a tangent vector τ. I leave it to the reader to check that the correspondence $\tau \leftrightarrow t$ is indeed linear. □

Theorem 3.3. *(Fogarty* [**10**]*) — Let X be a smooth connected quasiprojective surface. Then for each $n \in \mathbb{N}_0$, the Hilbert scheme* $\operatorname{Hilb}^n(X)$ *is connected and smooth of dimension $2n$.*

Proof. Let $[Z] \in \operatorname{Hilb}^n(X)$ be a closed point. We need to compute the dimension of $T_{[Z]} \operatorname{Hilb}^n(X) \cong \operatorname{Hom}(I_Z, \mathcal{O}_Z) = \operatorname{Hom}(I_Z/I_Z^2, \mathcal{O}_Z)$. Apparently, this is a completely local datum. We may therefore assume for simplicity that X is projective. From the long exact Ext-sequence associated to

$$0 \longrightarrow I_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0 \tag{6}$$

we read off that $\mathrm{Hom}(I_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$. Now

$$\mathrm{hom}(\mathcal{O}_Z, \mathcal{O}_Z) = h^0(\mathcal{O}_Z) = n, \tag{7}$$

and, by Serre duality,

$$\mathrm{ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) = \mathrm{hom}(\mathcal{O}_Z, \mathcal{O}_Z \otimes \omega_X) = h^0(\mathcal{O}_Z \otimes \omega_X) = n. \tag{8}$$

The theorem of Hirzebruch-Riemann-Roch yields

$$n - \mathrm{ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) + n = \int_X \mathrm{ch}(\mathcal{O}_Z)\mathrm{ch}(\mathcal{O}_Z)^\vee \mathrm{td}(X), \tag{9}$$

where $^\vee$ is the automorphism on H^{even} defined as $(-1)^i$ on H^{2i}. The integral is zero for dimension reasons, because Z has codimension 2. We conclude that $\dim_{\mathbb{C}} T_{[Z]} \operatorname{Hilb}^n(X) = 2n$ for all closed points $[Z]$.
Observe that $\operatorname{Hilb}^n(X)$ always contains the smooth $2n$-dimensional configuration space of unordered n-tuples of pairwise disjoint points as an open subscheme. Thus in order to see that $\operatorname{Hilb}^n(X)$ is smooth of dimension $2n$ it suffices to show that $\operatorname{Hilb}^n(X)$ is connected. We will later give a proof (see Lemma 3.7) by a method different from Fogarty's. □

Fogarty's theorem is the basis of all the nice properties of the Hilbert scheme of points on surfaces. For higher dimensional varieties much less is true:

Corollary 3.4. — *Let X be a smooth quasiprojective variety of dimension d and let $[Z] \in \operatorname{Hilb}^n(X)$ be a closed point such that $\dim T_x Z \leq 2$ for all $x \in Z$. Then $\operatorname{Hilb}^n(X)$ is smooth of dimension dn at $[Z]$. In particular, $\operatorname{Hilb}^n(X)$ is smooth for all d if $n \leq 3$.*

Proof. It suffices to consider the case that Z is supported in a single point $x \in X$. Let $d' := \dim T_x Z \leq d$. We may choose local coordinates $z_1, \ldots, z_d$ near x such that $z_{d'+1}, \ldots, z_d$ are contained in the ideal I of Z. Let $Y \subset X$ be the subvariety defined by $J := (z_{d'+1}, \ldots, z_d)$. Then Y is smooth at x and contains Z. The ideal of Z in Y is $\bar{I} := I/J$. There is an isomorphism of $\mathcal{O}_Z$-modules $I/I^2 \cong \mathbb{C}\langle \bar{z}_{d'+1}, \ldots, \bar{z}_d \rangle \otimes \mathcal{O}_Z \oplus \bar{I}/\bar{I}^2$. Therefore,

$$T_{[Z]} \operatorname{Hilb}^n(X) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\langle \bar{z}_{d'+1}, \ldots, \bar{z}_d \rangle, \mathcal{O}_Z) \oplus T_{[Z]} \operatorname{Hilb}^n(Y). \tag{10}$$

By assumption, $d' \leq 2$, hence $\operatorname{Hilb}^n(Y)$ is smooth at $[Z]$ of dimension $d'n$. We obtain: $\dim T_{[Z]} \operatorname{Hilb}^n(X) = (d - d') \cdot \ell(\mathcal{O}_Z) + d'n = dn$. This proves the first assertion. In general, one always has $d' \leq n - 1$. Hence the assumptions for the first claim are always satisfied if $n \leq 3$. This proves the second claim. □

Remark 3.5. — $\mathrm{Hilb}^4(\mathbb{C}^3)$ is singular at $[Z] = V(\mathfrak{m}^2)$, where $\mathfrak{m} = (x, y, z) \subset \mathbb{C}[x, y, z]$. For

$$T_{[Z]} \mathrm{Hilb}^n(\mathbb{C}^3) = \mathrm{Hom}_{\mathbb{C}[x,y,z]}(\mathfrak{m}^2/\mathfrak{m}^4, \mathbb{C}[x, y, z]/\mathfrak{m}^2) = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}^2/\mathfrak{m}^3, \mathfrak{m}/\mathfrak{m}^2) \tag{11}$$

has dimension 18, whereas $\mathrm{Hilb}^4(\mathbb{C}^3)$ is twelve-dimensional.

Let X be a smooth projective surface. The projection $p : \Xi_n \to \mathrm{Hilb}^n(X)$ is flat and finite of degree n. This implies that $\mathcal{A} := p_*\mathcal{O}_{\Xi_n}$ is a locally free sheaf of rank n. Moreover, $\mathcal{A}$ carries an algebra structure, and $\Xi_n \cong \underline{\mathrm{Spec}}(\mathcal{A})$.
Let $\mathcal{A} \to \mathcal{O}_{\mathrm{Hilb}^n(X)}$ denote the trace map for the action of $\mathcal{A}$ on itself. The composition with the multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a quadratic form $\mathcal{A} \otimes \mathcal{A} \to \mathcal{O}_{\mathrm{Hilb}^n(X)}$. Its discriminant vanishes in $[\xi]$ if and only if $\mathcal{A}([\xi]) = H^0(\mathcal{O}_\xi)$ is not semisimple, i.e. if ξ has multiplicity ≥ 2 at some of its points. The set

$$\partial \mathrm{Hilb}^n(X) := \{[\xi] \in \mathrm{Hilb}^n(X) \mid \exists\, x \in X :\ \mathrm{length}(\mathcal{O}_{\xi,x}) \geq 2\}. \tag{12}$$

is called the boundary of the Hilbert scheme. It is the vanishing locus of the homomorphism $\mathcal{A} \longrightarrow \mathcal{A}^*$. In particular, it is a divisor satisfying

$$[\partial \mathrm{Hilb}^n(X)] = -2c_1(\mathcal{A}). \tag{13}$$

3.2. The Hilbert-Chow-morphism. Let X be a reduced quasiprojective variety. To any closed point $[Z] \in \mathrm{Hilb}^n(X)$ we may associate a formal sum $\sum_{x\in X} \ell(\mathcal{O}_{Z,x}) \cdot x$, i.e. closed point in the symmetric product $S^n(X)$. This defines the Hilbert-Chow map

$$\rho : \mathrm{Hilb}^n(X) \longrightarrow S^n(X), \tag{14}$$

at least set-theoretically. That ρ is indeed a morphism can be seen as follows (cf. [**19**]): suppose $Z \in \underline{\mathrm{Hilb}}^n(X)(S)$, and let $s_0 \in S$ be a closed point corresponding to a subscheme $Z_0 \subset X$. Since X is quasiprojective, there is an open affine subscheme $U = \mathrm{Spec}\ A \subset X$ that contains Z_0. As the projection $p : Z \to S$ is proper, $p(S \times (X \setminus U) \cap Z)$ is a closed subset in S. Its complement is an open subset V' with the property that $Z_s \subset U$ for all $s \in V'$. Let $V = \mathrm{Spec}\ B$ be an open affine neighbourhood of s_0 in V'. Now $Z_V := Z \cap V \times U$ is affine, say $Z_V \cong \mathrm{Spec}\ C$, where C is a factor ring of $B \otimes A$. Moreover, C is a projective B-module of rank n, since $p : Z_V \to V$ is flat and finite of degree n. The natural surjection $B \otimes A \to C$ induces a ring homomorphism $f : A \to \mathrm{End}_B(C)$. There is an induced action of $A^{\otimes n}$ on $C^{\otimes n}$, the tensor products being taken over $\mathbb{C}$ and B, respectively. The subring of invariant tensor $S_nA := (A^{\otimes n})^{S_n}$ acts on the rank 1 submodule $\Lambda_nC \subset C^{\otimes n}$ of antisymmetric tensors. This yields a ring homomorphism $\varphi : S_nA \to \mathrm{End}_B(\Lambda_nC) =$

B and thus a morphism

$$V = \text{Spec } B \longrightarrow S^n U := \text{Spec}((A^{\otimes n})^{S_n}) \subset S^n X. \tag{15}$$

The construction is functorial in V and therefore yields a morphism

$$\rho : \text{Hilb}^n(X) \to S^n X. \tag{16}$$

Explicitly the ring homomorphism $\varphi : S_n A \to B$ can be described as follows: Let $ld : \text{End}_B(C)^{\otimes n} \to B$ be the polar form of the determinant, i.e.

$$\text{ld}(e_1, \ldots, e_n) := \frac{1}{n!}\text{coeff}\big(t_1 \cdot \ldots \cdot t_n, \det(t_1 e_1 + \cdots + t_n e_n)\big), \tag{17}$$

Then $\varphi(a_1 \otimes \ldots \otimes a_n) = \text{ld}(f(a_1), \ldots, f(a_n))$. We must show that for a $\mathbb{C}$-valued point $[Z]$ of $\text{Hilb}^n(X)$ one has $\rho([Z]) = \sum_{x \in X} \ell(\mathcal{O}_{Z,x}) \cdot x$. The situation above simplifies: $B = \mathbb{C}$ and C is an artinian factor algebra of A of length n. The ring homomorphism φ factors through $S_n A \to S_n C$ and a homomorphism $S_n C \to \text{End}_{\mathbb{C}}(\Lambda_n C)$. Decompose $C = C_1 \times \ldots \times C_s$ with local artinian rings C_i of length n_i respectively. Then $\Lambda_n C \cong \bigotimes_i \Lambda_{n_i} C_i$ and $S_n C = \prod_\ell \bigotimes_i S_{\ell_i} C_i$, where the product runs over decompositions $n = \ell_1 + \ldots + \ell_s$. All of these factors act trivially on $\Lambda_n C$ except the one corresponding to the decomposition $\ell_i = n_i$. Thus φ factors as follows:

$$S_n A \longrightarrow S_n C \longrightarrow \bigotimes_i S_{n_i} C_i \longrightarrow \text{End}_{\mathbb{C}}(\Lambda_n C) = \mathbb{C}. \tag{18}$$

This shows that the closed point $x_i = (\text{Spec } C_i)_{\text{red}}$ appears in $\rho([Z])$ with multiplicity n_i.

Definition 3.6. — Let X be a smooth quasiprojective surface, and let $\Delta := \{nx \mid x \in X\} \subset S^n X$ denote the small diagonal. Let $B^n := \rho^{-1}(\Delta)$ and $B^n_p := \rho^{-1}(np)$ for some $p \in X$, both subsets equipped with the reduced induced subscheme structure.

We will refer to B^n_p as the n-th Briançon variety. Any étale morphism $f : X \to X'$ induces an isomorphism $B^n_p \to B^n_{f(p)}$. Thus all Briançon varieties are non-canonically isomorphic, regardless which surface X or which point $p \in X$ one considers. In fact one can show that $B \to \Delta$ is a fibre bundle in the Zariski topology.

3.3. An induction scheme. A basic tool to study Hilbert schemes of points on a quasiprojective scheme X is an induction procedure that allows one to compare the geometric properties of $\text{Hilb}^n(X)$ and $\text{Hilb}^{n+1}(X)$.

Let $Z \subset \text{Hilb}^n(X) \times \text{Hilb}^{n+1}(X)$ denote the subscheme whose closed points consist of pairs (ξ, ξ') such that ξ is a subscheme of ξ'. The scheme structure of Z is defined by the vanishing of the homomorphism

$$\mathrm{pr}_{23}^*(I_{\Xi_{n+1}}) \longrightarrow \mathcal{O}_{\mathrm{Hilb}^n(X)\times\mathrm{Hilb}^{n+1}(X)\times X} \longrightarrow \mathrm{pr}_{13}^*(\mathcal{O}_{\Xi_n}). \tag{19}$$

The subscheme Z comes with two projections:

$$\mathrm{Hilb}^n(X) \xleftarrow{\ \varphi\ } Z \xrightarrow{\ \psi\ } \mathrm{Hilb}^{n+1}(X). \tag{20}$$

In order to make use of this diagram we need a more specific description of φ and ψ. We follow here a method due to Ellingsrud and Strømme [**8**].
Suppose that $\xi \subset \xi' \subset X$ are subschemes of length n and $n+1$, respectively. Then there are short exact sequences

$$0 \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_{\xi'} \longrightarrow \mathcal{O}_\xi \longrightarrow 0 \tag{21}$$

and

$$0 \longrightarrow I_{\xi'} \longrightarrow I_\xi \longrightarrow \mathcal{O}_x \longrightarrow 0 \tag{22}$$

where $\mathcal{O}_x$ denotes the structure sheaf of the reduced point $x \in X$, the point where ξ and ξ' differ. It is apparent from (22) that we may pass from ξ to ξ' by choosing a point $x \in X$ and a surjective homomorphism $\lambda : I_\xi(x) = I_\xi/\mathfrak{m}_x I_\xi \longrightarrow \mathbb{C}$. Any ξ' can be obtained in this way, and ξ' determines λ up to a scalar. More formally, let $\Phi = (\phi, \rho) : \mathbb{P} := \mathbb{P}(I_{\Xi_n}) \to \mathrm{Hilb}^n(X) \times X$ denote the projectivisation of I_{Ξ_n}. We will construct a flat family of subscheme in X of length $n+1$ and parameterised by $\mathbb{P}$ as follows: There is a canonical surjection

$$\alpha : \Phi^*(I_{\Xi_n}) \longrightarrow \mathcal{O}_{\mathbb{P}}(1). \tag{23}$$

Let $Y \subset \mathbb{P} \times X$ denote the preimage of the diagonal $\Delta_X \subset X \times X$ under the morphism $\rho \times \mathrm{id}_X : \mathbb{P} \times X \to X$. Consider the surjective homomorphism of sheaves on $\mathbb{P} \times X$:

$$\beta : (\phi\times\mathrm{id}_X)^* I_{\Xi_n} \longrightarrow (\phi\times\mathrm{id}_X)^* I_{\Xi_n}|_Y \cong (\mathrm{pr}_1^*\, \Phi^* I_{\Xi_n})|_Y \xrightarrow{\mathrm{pr}_1^*\alpha} (\mathrm{pr}_1^*\, \mathcal{O}_{\mathbb{P}}(1))|_Y. \tag{24}$$

Its kernel is the ideal sheaf of a family of subschemes of length $n+1$. Hence there is classifying morphism $\psi : \mathbb{P} \to \mathrm{Hilb}^{n+1}(X)$ such that $\ker(\beta) \cong (\psi \times \mathrm{id}_X)^* I_{\Xi_{n+1}}$. We obtain a short exact sequence

$$0 \longrightarrow (\psi \times \mathrm{id}_X)^* I_{\Xi_{n+1}} \longrightarrow (\phi \times \mathrm{id}_X)^* I_{\Xi_n} \longrightarrow \mathrm{pr}_1^*\, \mathcal{O}_{\mathbb{P}}(1)|_Y \longrightarrow 0 \tag{25}$$

of sheaves on $\mathbb{P} \times X$. This is the global version of sequence (22). The morphism $(\phi, \psi) : \mathbb{P} \to \mathrm{Hilb}^n(X) \times \mathrm{Hilb}^{n+1}(X)$ is an isomorphism onto the subscheme Z.

We also note that $\mathbb{P}$ comes with a map $\rho : \mathbb{P} \to X$ that sends $\xi, \xi')$ to the point x, where ξ is modified. The induction scheme takes the final form

$$\begin{tikzcd} \mathrm{Hilb}^n(X) & \mathbb{P} \arrow[l, "\phi"'] \arrow[r, "\psi"] \arrow[d, "\rho"] & \mathrm{Hilb}^{n+1}(X) \\ & X & \end{tikzcd} \tag{26}$$

Here is a first application of this construction. The following lemma fills the remaining gap in the proof of Fogarty's theorem 3.3.

Lemma 3.7. — *Let X be a connected quasiprojective scheme. Then* $\mathrm{Hilb}^n(X)$ *is connected for all* $n \geq 0$.

Proof. $\mathrm{Hilb}^0(X)$ is a point, $\mathrm{Hilb}^n(X) = X_{\mathrm{red}}$ is connected. By induction we may assume that $\mathrm{Hilb}^n(X)$ is known to be connected. The fibres of the map $\Phi := (\phi, \rho) : \mathbb{P} \to \mathrm{Hilb}^n(X) \times X$ are projective spaces and hence connected. Thus $\mathbb{P}$ is connected. As ψ is surjective, $\mathrm{Hilb}^{n+1}(X)$ is connected as well. □

The symmetry suggested by diagram (26) is not quite apparent from the construction of $\mathbb{P}$ given above. However, if X is a smooth surface the following holds: The dualising sheaf $\omega_\xi := \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{O}_\xi, \omega_X) = \mathcal{E}xt^1(I_\xi, \omega)$ of ξ is an $\mathcal{O}_\xi$-sheaf of length n. Applying $\mathcal{H}om(-, \omega_X)$ to sequence (21) we obtain an exact sequence

$$0 \longrightarrow \omega_\xi \longrightarrow \omega_{\xi'} \longrightarrow \mathcal{O}_x \longrightarrow 0. \tag{27}$$

Thus to pass from ξ' to ξ, we have to choose a closed point x in the support of ξ' and a surjective homomorphism $\mu : \omega_{\xi'}(x) \to \mathbb{C}$. This defines a point $[\mu] \in \mathbb{P}(\omega_{\xi'}(x))$. Now we can argue as above and show that the morphism $\Psi : (\psi, \rho) : \mathbb{P} \to \mathrm{Hilb}^{n+1}(X) \times X$ is isomorphic to the canonical projection $\mathbb{P}(\omega_{\Xi_{n+1}}) \to \mathrm{Hilb}^{n+1}(X) \times X$, where $\omega_{\Xi_{n+1}} = \mathcal{E}xt^2(\mathcal{O}_{\Xi_{n+1}}, \omega_X)$ is the relative dualising sheaf of the universal family.

From now on let X denote a smooth irreducible projective surface. Let $\mathbb{P}'$ denote the blow-up of $\mathrm{Hilb}^n(X) \times X$ along the subscheme Ξ_n, and let $E \subset \mathbb{P}'$ be the exceptional divisor. The ring epimorphism $S^*(I_{\Xi_n}) \twoheadrightarrow \bigoplus_{\nu \geq 0} I^\nu_{\Xi_n}$ induces a closed embedding $\mathbb{P}' \to \mathbb{P}$.

Proposition 3.8. — *$\mathbb{P}$ is irreducible and hence isomorphic to $\mathbb{P}'$.*

Proof. Let H be an ample divisor on X. For sufficiently large m, the sheaf $B := p^*\big(p_*(\mathcal{O}_{\Xi_n}(mH)\big)(-mH)$ is locally free, and the evaluation map $B \to I_{\Xi_n}$ is surjective. Let A denote its kernel. Since I_{Ξ_n} is flat over $\mathrm{Hilb}^n(X)$, the same is true for A and B. Hence for each $[\xi] \in \mathrm{Hilb}^n(X)$, the sequence

$$0 \longrightarrow A|_{\{\xi\}\times X} \longrightarrow B|_{\{\xi\}\times X} \longrightarrow I_\xi \longrightarrow 0 \tag{28}$$

is still exact. Since X is smooth of dimension 2, the global dimension of $\mathcal{O}_\xi$ is ≤ 2 and that of I_ξ is ≤ 1. It follows that $A|_{\{\xi\}\times X}$ is locally free. By flatness, A is locally free, too. We obtain a global resolution

$$0 \longrightarrow A \longrightarrow B \longrightarrow I_{\Xi_n} \longrightarrow 0 \tag{29}$$

with locally free sheaves A and B of rank a and $b = a + 1$, respectively. The epimorphism $B \to I_{\Xi_n}$ induces a closed embedding $\mathbb{P} \to \mathbb{P}(B)$, and the image is the zero set of the composite map $\Phi^*A \to \Phi^*B \to \mathcal{O}_B(1)$. Thus $\mathbb{P}$ is locally cut out from the smooth variety $\mathbb{P}(B)$ by a equations. It follows that every irreducible component of $\mathbb{P}$ has dimension $\geq \dim(\mathrm{Hilb}^n(X) \times X) + \mathrm{rk}(B) - 1 - \mathrm{rk}(A) = 2n+2$. Let $p \in \mathbb{P}$ and let $[\xi] = \phi(p)$, $[\xi'] = \psi(p)$, $x = \rho(p)$ denote the images of p in $\mathrm{Hilb}^n(X)$, $\mathrm{Hilb}^{n+1}(X)$ and X, respectively. The corresponding ideal sheaves, structure sheaves and dualising sheaves are related by the sequences (21), (22) and (27). The invariants $i := i(\xi, x) := \dim I_\xi(x)$, $i' := i(\xi', x)$ and $j' := j(\xi', x) := \dim \omega_{\xi'}(x)$ are related by

$$|i - i'| \leq 1, \quad j' = i' - 1. \tag{30}$$

This can be seen as follows: Tensoring (22) by $\mathcal{O}_x$, we obtain

$$\mathbb{C}^2 = \mathrm{Tor}_1(\mathcal{O}_x, \mathcal{O}_x) \longrightarrow I_{\xi'}(x) \longrightarrow I_\xi(x) \longrightarrow \mathbb{C} \longrightarrow 0 \tag{31}$$

and conclude that $-1 \leq i - i' \leq 1$. To prove the second assertion, choose a locally free resolution $0 \to A' \to B' \to I_{\xi'} \to 0$. Then $I_{\xi'}(x) = \mathrm{coker}(A'(x) \xrightarrow{m} B'(x))$ and $\omega_{\xi'}(x) = \mathrm{coker}(B'(X)^* \xrightarrow{m^*} A'(X)^*)$. It follows that

$$j' = \mathrm{rk}(A') - \mathrm{rk}(m^*) = (\mathrm{rk}(B') - 1) - \mathrm{rk}(m) = i' - 1. \tag{32}$$

We stratify $\mathrm{Hilb}^n(X) \times X$ according to the fibre dimension of Φ:

$$\mathrm{Hilb}^n(X) \times X = \coprod_{i \geq 1} H_{n,i} \quad , \text{ with } H_{n,i} := \{(\xi, x) \mid i(\xi, x) = i\} \tag{33}$$

Then $\Phi^{-1}(\xi, x) \cong \mathbb{P}^{i-1}$ for $(\xi, x) \in H_{n,i}$, and similarly, $\Psi^{-1}(\xi', x) \cong \mathbb{P}^{j'-1} = \mathbb{P}^{i'-2}$ for $(\xi', x) \in H_{n+1,i'}$. Moreover, because of $|i - i'| \leq 1$, the inclusions

$$\Psi^{-1}(H_{n+1,i'}) \subset \bigcup_{|i-i'|\leq 1} \Phi^{-1}(H_{n,i}) \tag{34}$$

hold. This gives the dimension estimate

$$\dim H_{n+1,i'} + (i' - 2) \leq \max\{\dim H_{n,i} + (i - 1) \mid |i - i'| \leq 1\}. \tag{35}$$

Using this estimate one can prove the following bound by induction on n:

$$\dim H_{n,i} \leq 2n - 2(i-2) \quad \text{for all } i \geq 1 \tag{36}$$

The assertion is obvious for $i = 1$ and is also true for $i = 2$, since $H_{n,2} \subset \Xi_n$, the latter projecting finitely onto $\mathrm{Hilb}^n(X)$. Assume by induction that the assertion is true for n and consider $H_{n+1,i'}$. It follows from the inequality (35) that

$$\begin{aligned} \dim H_{n+1,i'} &\leq (i'-2) + \max\{2n - 2(i-2) + (i-1) \mid -1 \leq i - i' \leq 1\} \\ &= 2(n+1) - 2(i'-2). \end{aligned} \tag{37}$$

This finishes the induction.

Using (36), we obtain $\Phi^{-1}(H_{n,i}) \leq 2n + 2 - (i-1) < 2n + 2$ for $i \geq 2$. Since any irreducible component of $\mathbb{P}$ must have dimension $\geq 2n+2$, $\mathbb{P}$ must be contained in the closure of $\Phi^{-1}(H_{n,1})$. This implies that $\mathbb{P}$ is irreducible and therefore isomorphic to the blow-up $\mathbb{P}'$. $\square$

Proposition 3.9. — *The divisor $\partial\,\mathrm{Hilb}^n(X) \subset \mathrm{Hilb}^n(X)$ is irreducible.*

Proof. We will show that $\partial\,\mathrm{Hilb}^n(X)$ is irreducible by induction on n. Clearly, $\partial\,\mathrm{Hilb}^1(X)$ is empty, and $\partial\,\mathrm{Hilb}^2(X) \cong X$. We will make use of the notation and construction introduced in the proof of Proposition 3.8:

the exceptional divisor $E \subset \mathbb{P}' = \mathbb{P}$ equals $\bigcup_{i\geq 2}\Phi^{-1}(H_{n,i})$. Every component of E has dimension $\geq 2n+1$. Since $\dim \Phi^{-1}(H_{n,i}) < 2n+1$ for $i \geq 3$, it follows that $\Phi^{-1}(H_{n,2})$ is dense in E. Since the map $H_{n,2} \mapsto \partial\mathrm{Hilb}^n(X)$ is birational, we may assume by induction that E is irreducible. Since $\partial\,\mathrm{Hilb}^n(X) = \psi(E)$ this proves the claim. $\square$

Theorem 3.10. *(Briançon* [**1**]*) — Let X be a smooth projective surface and let $p \in X$ be a closed point. The varieties B_p^n are irreducible and of dimension $n-1$. In particular, they contain the set of curvilinear subschemes as an open dense subset.*

Proof. We proceed by induction and assume that the theorem holds for n. We may assume that X is irreducible and continue to use the construction and notation introduced above. The stratification (33) of $\mathrm{Hilb}^n(X) \times X$ induces a stratification

$$B_p^n \times \{p\} = \coprod_{i\geq 2} B_{n,i} \quad , \text{ with } B_{n,i} = B_p^n \times \{p\} \cap H_{n,i}. \tag{38}$$

Now $\Phi^{-1}(B_p^n \times \{p\}) = E \cap \Phi^{-1}(B_p^n \times X) =: E'$ is a Cartier divisor in the blow-up of $B_p^n \times X$ along $\Xi_n \cap B_p^n \times X$. Since the blow-up has dimension $n+1$, every irreducible component of E' has dimension $\geq n$. We will prove the following estimate by induction:

$$\dim B_{n,i} \leq (n-1) - (i-1) \quad \text{for } i \geq 3. \tag{39}$$

Assuming this estimate for n, it follows that $\dim \Phi^{-1}(B_{n,i}) \leq (n-i)+(i-1) = n-1$ for all $i \geq 3$. This shows that E' is contained in the closure of $\Phi^{-1}(B_{n,2})$ and hence irreducible. It follows that $B_p^{n+1} = \psi(E')$ is irreducible, too. Moreover, since $\psi : E' \to B_p^{n+1}$ is generically an isomorphism, $\dim(B_p^{n+1}) \leq \dim(B_p^n) + 1 = n$.
It remains to prove (39). We argue as before:

$$\dim B_{n+1,i'} + (i'-2) \leq \max\{\dim B_{n,i} + (i-1) \mid |i-i'| \leq 1\}. \tag{40}$$

For $i' \geq 4$, we have always $i \geq 3$ and conclude by induction that $\dim B_{n+1,i'} \leq (n+1) - i'$. If $i' = 3$, we cannot exclude that the case $i = 2$ appears. However, $\Psi^{-1}(B_{n+1,2}) \cap \Phi^{-1}(B_{n,2})$ must be a proper subset in $\Phi^{-1}(B_{n,2})$ and hence have dimension $< n$. We get the desired estimate for $i' = 3$ as well. □

4. The Cohomology of $\mathrm{Hilb}^n(X)$

Let X be a smooth irreducible projective surface. By Fogarty's theorem, the Hilbert scheme $\mathrm{Hilb}^n(X)$ is a compact complex manifold of dimension $2n$. This section deals with the computation of the cohomology of the Hilbert scheme and, most important, the additional structure as a module over a certain infinite dimensional Lie algebra as defined by Nakajima. In the following, all cohomology groups of varieties refer to cohomology with coefficients in $\mathbb{Q}$ (or any other field of characteristic 0).

4.1. Göttsche's Formula. Since $\mathrm{Hilb}^n(X)$ is a manifold of complex dimension $2n$, its rational cohomology groups range from degree 0 to degree $4n$. The Betti numbers were computed by Göttsche [**14**] who gave the following formula that expresses the Betti numbers of $\mathrm{Hilb}^n(X)$ in terms of the Betti numbers of X:

Theorem 4.1. *(Göttsche [**14**])* —

$$\sum_{n=0}^{\infty}\sum_{i=0}^{4n} b_i(\mathrm{Hilb}^n(X))p^{i-2n}q^n = \prod_{m=1}^{\infty}\prod_{j=0}^{4}(1-(-1)^j p^{j-2}q^m)^{-(-1)^j b_j(X)}. \tag{41}$$

Here t and q are formal parameters. One of the consequences of this formula is that the Betti numbers stabilise, i.e. for fixed i and $n \to \infty$, $b_i(\mathrm{Hilb}^n(X))$ becomes constant.
It also becomes clear that one should not expect to get nice formulae for a single Hilbert scheme, but that one should rather look at all Hilbert schemes simultaneously. Let us introduce the double graded vector space

$$\mathbb{H} := \bigoplus_n \mathbb{H}^n = \bigoplus_{n,i} \mathbb{H}^{n,i}, \quad \mathbb{H}^{n,i} := H^{i-2n}(\mathrm{Hilb}^n(X)). \tag{42}$$

The right hand side in Göttsche's formula is also the Poincaré series of an irreducible module over the so-called Heisenberg Lie algebra. I will explain this in the next section. As a warm-up consider the following toy example.

EXAMPLE 4.2. — Consider the Lie algebra $\mathfrak{g}_{\text{even}}$ spanned by elements p, q, c that are subject to the relations

$$[p,q]=c, \quad [p,c]=0, \quad [q,c]=0, \tag{43}$$

so that c is a central element. $\mathfrak{g}_{\text{even}}$ acts on the vector space $M := \mathbb{C}[x]$ via $p \mapsto \frac{\partial}{\partial x}$, $q \mapsto x\cdot$ and $c \mapsto \mathrm{id}$. The Leibniz rule for the differential implies that this is indeed a represensation of $\mathfrak{g}$. Moreover, M is an irreducible representation: If $V \subset M$ is a nonzero submodule, and if $f = f_n x^n + \ldots + f_0 \in V$ a polynomial of degree n, then V also contains $1 = \frac{1}{n! f_n} p^n(f)$. Applying q sufficiently often, we see that V contains all powers of x and therefore coincides with M.
The module M is naturally graded, and its Poincaré series is

$$\sum_{n\geq 0} \dim M_n\, t^n = \sum_{n\geq 0} t^n = (1-t)^{-1}. \tag{44}$$

Next, we modify the example slightly: We now assume that p and q are odd elements. The commutator has to be redefined as $[p,q] = pq + qp$. We keep c even and take the same relations as before. (However, we have to add $[p,p] = 0 = [q,q]$, since these relations are no longer automatic.) The super Lie algebra $\mathfrak{g}_{\text{odd}}$ spanned by p, q and c acts on the vector space underlying the exterior algebra $M = \mathbb{C} \oplus \mathbb{C}x$ of a one dimensional vector space $\mathbb{C}\langle x\rangle$ rather than the symmetric algebra as before. The action is given by $p \mapsto \frac{\partial}{\partial x}$, $q \mapsto x\wedge$ and $c \mapsto \mathrm{id}$. The Poincaré series of M is $(1+t)$.
Both Poincaré series can be uniformly written as $(1-\varepsilon t)^{-\varepsilon}$ with $\varepsilon = \pm 1$ according to whether p and q are even or odd elements.

If we now look at Göttsche's formula again, we realise that the factors on the right hand side have a very similar structure as the Poincaré series in the example. But instead of one factor there are infinitely many: this indicates that one deals with an infinite tensor product of modules such as M. And instead of one formal parameter t there are two, p and q, indicating that the graded modules are in fact bigraded.
Nakajima constructed an action of an infinite dimensional Lie algebra on $\mathbb{H}$ that realises all these features. We will first discuss the abstract algebraic construction of the Lie algebra and then Nakajima's geometric representation of it.

4.2. The Heisenberg Lie algebra and an irreducible representation. By definition, a super vector space is a vector space H together with a $\mathbb{Z}/2$-grading $H = H^{\text{even}} \oplus H^{\text{odd}}$. If H and H' are super vector spaces, $\mathrm{Hom}(H, H')$ and $H \otimes H'$ etc. inherit $\mathbb{Z}/2$-gradings in a natural way. The symmetric group S_n acts on $H^{\otimes n}$ by permuting the factors of a product of homogeneous elements with an additional sign:

$$\pi(h_1 \otimes \ldots \otimes h_n) = (-1)^\varepsilon h_{\pi^{-1}(1)} \otimes \ldots \otimes h_{\pi^{-1}(n)}, \tag{45}$$

where $\varepsilon = \sum_{i<j, \pi(i)>\pi(j)} |h_i| \cdot |h_j|$ and $|h_i| \in \mathbb{Z}/2$ denotes the parity of h_i. This affects the definition of the symmetric and the exterior algebras of a super vector space. As a consequence

$$S^*(H^{\mathrm{even}} \oplus H^{\mathrm{odd}}) = S^*(H^{\mathrm{even}}) \otimes \Lambda^*(\Pi H^{\mathrm{odd}}). \tag{46}$$

Here Π is the parity change functor: it makes H^{odd} even and H^{even} odd.
Let H be finite-dimensional super vector space over $\mathbb{Q}$ with a non-degenerate even symmetric bilinear form $\langle -, - \rangle : H \times H \to \mathbb{Q}$. Evenness of the form means that $H^{\mathrm{even}} \perp H^{\mathrm{odd}}$, and symmetry means that

$$\langle \alpha, \beta \rangle = (-1)^{|\alpha| \cdot |\beta|} \langle \beta, \alpha \rangle \tag{47}$$

for homogeneous elements of degree $|\alpha|, |\beta| \in \mathbb{Z}/2$.
A typical example for such a situation is the cohomogy ring $H := H^*(X, \mathbb{Q})$ of a compact manifold X of even real dimension. The pairing in this example is the cup product pairing $\int_X \alpha \cup \beta$. The cup product itself is not needed for the moment.
To the datum $(H, \langle -, - \rangle)$ we associate a so-called Heisenberg Lie algebra as follows: Define a Lie bracket on

$$\mathfrak{h} := H[t, t^{-1}] \oplus \mathbb{Q}c \tag{48}$$

by $[c, u] := 0$ for all $u \in \mathfrak{h}$ and

$$[\alpha f(t), \beta g(t)] := \langle \alpha, \beta \rangle \ \mathrm{res}_t\, f\, dg \cdot c \tag{49}$$

for all $\alpha, \beta \in H$, $f, g \in \mathbb{C}[t, t^{-1}]$. This bracket is super skew-symmetric, since the residue of $d(fg)$ vanishes so that the Leibniz rule implies that $\mathrm{res}_t\, f\, dg = -\,\mathrm{res}_t\, g\, df$. Since c is central by definition, the bracket satisfies the Jacobi-identity.
It is convenient to introduce the variables $\alpha_n := \alpha\, t^n$ for $\alpha \in H$. Then the relations can be written as $[\alpha_n, c] = 0$ and

$$[\alpha_n, \beta_m] = n\, \delta_{n,-m} \langle \alpha, \beta \rangle\, c. \tag{50}$$

Note that here and elsewhere whenever we are concerned with possibly odd elements x and y, the Lie bracket of x and y will be super skew symmetric, i.e.

$$[x, y] = -(-1)^{|x| \cdot |y|} [y, x]. \tag{51}$$

If we choose a homogeneneous basis α^i of H with dual basis β^i, then each pair α^i_n, β^i_{-n} spans a Lie algebra isomorphic to either $\mathfrak{g}_{\mathrm{even}}$ or $\mathfrak{g}_{\mathrm{odd}}$ in example 4.2 depending on whether α is an even or an odd element.

The next step consists in constructing a highest (or rather: lowest) weight representation of $\mathfrak{h}$. Let $I \subset U(\mathfrak{h})$ be the left ideal in the universal envelopping algebra of $\mathfrak{h}$ that is generated by all elements of the form α_n with $n \geq 0$, and the element $c - 1$.

Then the quotient $V := U(\mathfrak{h})/I$ is as a representation of $\mathfrak{h}$. It is generated as a representation by the residue class $\mathbf{1}$ of $1 \in U(\mathfrak{h})$. By construction, we have

$$\alpha_n \mathbf{1} = 0, \text{ for all } n \geq 0, \text{ and } c.\mathbf{1} = \mathbf{1}. \tag{52}$$

We will refer to $\mathbf{1}$ as the vacuum element. Let $\mathfrak{h}_- = t^{-1}H[t^{-1}]$. The Poincaré-Birkhoff-Witt theorem implies that V is isomorphic as a vector space to $U(\mathfrak{h}_-) \cong S(\mathfrak{h}_-)$.

A good way of looking at V is the following. Using the basis α^i of H and its dual basis b^i as above, V may be considered as the underlying vector space of the (super) polynomial ring in the variables α^i_{-n}, $i = 1, \ldots, \dim(H)$, $n \in \mathbb{N}$. On these monomials the action of $\mathfrak{h}$ is given as follows: c acts as the identity, α^j_{-m} by multiplication with α^j_{-m}, and β^j_m acts as $m\frac{\partial}{\partial \alpha^j_{-m}}$ for $m \in \mathbb{N}_0$. The same reasoning as in example 4.2 shows:

Lemma 4.3. — *V is an irreducible $\mathfrak{h}$-module*

The Heisenberg Lie algebra can be endowed with an additional $\mathbb{Z}$-grading by *weight*. We declare $\mathrm{wt}(\alpha_n) := -n$ and $\mathrm{wt}(c) = 0$. It is easy to check that the Lie bracket is homogeneous, and also that the ideal $I \subset U(\mathfrak{h})$ is homogeneous. Hence $V = U(\mathfrak{h})/I$ inherits a grading, and the action of $\mathfrak{h}$ on V is homogeneous. In order to compute its Poincaré series, all we need is to remember the fact that $S^*(\bigoplus W_i) = \bigotimes S^*(W_i)$ for any family of vector spaces. Therefore there are isomorphisms of graded vector spaces, where I have put in a formal variable to indicate the weight of the appearing vector spaces:

$$\bigoplus_{n \geq 0} V_n q^n = S^* \left(\bigoplus_{m>0} Hq^m \right) = \bigotimes_{m>0} S^*(H^{\mathrm{even}} q^m) \otimes \bigotimes_{m>0} \Lambda^*(H^{\mathrm{odd}} q^m). \tag{53}$$

Passing to dimensions, we obtain the following formula for the Poincaré series:

$$\sum_{n \geq 0} \dim(V^n) q^n = \prod_{m>0} \frac{(1+q^m)^{\dim H^{\mathrm{odd}}}}{(1-q^m)^{\dim H^{\mathrm{even}}}}. \tag{54}$$

Assume now in addition that the $\mathbb{Z}/2$–grading of H comes from a $\mathbb{Z}$–grading $H = \bigoplus_i H^i$. We will denote the degree of a homogeneous element $\alpha \in H$ as $|\alpha| \in \mathbb{Z}$. Note that this reinterpretation of the symbol $|\alpha|$ does not affect any of the formulae above. Then V becomes a bigraded vector space, $V = \bigoplus_{n,i} V^{n,i}$, the bidegree of α_{-n} being $(n, |\alpha|)$.

A straight-forward extension of the calculation just completed gives

$$\sum_{n\geq 0}\sum_i \dim(V^{n,i})q^np^i = \prod_{m>0}\prod_j (1-(-1)^jp^jq^m)^{-(-1)^j\dim H^j}. \tag{55}$$

Apply the construction above to the following setting: Let X be a smooth projective surface and let $H := H^*(X,\mathbb{Q})[-2]$. Let $\langle\alpha,\beta\rangle := -\int_X \alpha\cup\beta$. (By definition, the pairing is zero if the degrees of α and β don't match. The additional sign was thrown in in order to keep future formulae free from signs.) Let $\mathfrak{h}$ be the Heisenberg Lie algebra constructed from $(H,\langle -,-\rangle)$ and let $\mathbb{V}$ denote the representation constructed above. Then $\mathbb{V}$ has the same Poincaré series as the joint cohomology $\mathbb{H}$ of all Hilbert schemes. In particular, $\mathbb{V}$ and $\mathbb{H}$ are isomorphic as bigraded vector spaces. Of course, this in itself doesn't mean much. The question arises: Is there a natural isomorphism $\mathbb{V}\xrightarrow{\cong}\mathbb{H}$? This was answered positively by Nakajima [**32**] and Grojnowski [**17**].

4.3. Nakajima's operators. As before X is a smooth irreducible projective surface, $H := H^*(X,\mathbb{Q})[-2]$ and $\mathbb{H} := \bigoplus H^*(\mathrm{Hilb}^n(X,\mathbb{Q})[-2n])$. To keep formulae and diagrams readable, we will denote the Hilbert scheme by $X^{[n]} := \mathrm{Hilb}^n(X)$. Let $n,\ell\geq 0$. Inside the triple product $X^{[\ell+n]}\times X\times X^{[\ell]}$ consider the incidence variety

$$Z = Z^{\ell,n+\ell} = \{(\xi',x,\xi)\mid \xi\subset\xi',\rho(\xi')=\rho(\xi)+nx\}. \tag{56}$$

Thus Z is made up from all triples where ξ is a subscheme of ξ', and ξ and ξ' differ only at the point x. Apparently, Z generalises the incidence variety from section 3.3.

We may decompose Z into strata $Z_0\cup\ldots\cup Z_\ell$ with Z_i consisting of all triples such that $h^0(\mathcal{O}_{\xi,x})=i$.

If x and ξ have disjoint support, $\xi'=\xi\cup\eta$ for some $\eta\in X^{[n]}$ with support in x. In fact, Z_0 is isomorphic to an open subscheme in $X^{[\ell]}\times B^n$, where B^n is the $(n+1)$–dimensional Briançon scheme introduced in section 3.2. Similarly, Z_1 is isomorphic to an open subscheme in $X^{[\ell-1]}\times B^{n+1}$. It follows that $\dim(Z_0)=2\ell+n+1$ and $\dim(Z_1)=2\ell+n$. With more care one can show that all other strata have dimension $<2\ell+n$. To the best of my knowledge, we still do not know whether Z is irreducible or not. However, Z_1 is in the closure of Z_0, and certainly Z has no other component of dimension $\geq 2\ell+n$. This suffices for the present purpose. The fundamental class $[Z]$ is an element of degree $2\ell+n+1$ in the Chow group of $X^{[\ell+n]}\times X\times X^{[\ell]}$.

Following Nakajima [**32**], we define an operator α_{-n} on $\mathbb{H}$ with $\alpha_{-n}:\mathbb{H}^\ell\to\mathbb{H}^{\ell+n}$ for any $\alpha\in H$ and $n\in\mathbb{N}$ as follows. Let p_i denote the projection from $X^{[\ell]}\times X\times X^{[\ell+n]}$

onto its i-th factor, $i = 1, 2, 3$. For $y \in \mathbb{H}^\ell = H^*(X^{[\ell]})[-2\ell]$ let

$$\alpha_{-n}(y) := PD^{-1} p_{1*}((p_2^*(\alpha) \cup p_3^*(y)) \cap [Z]), \tag{57}$$

where PD denotes Poincaré duality. It is clear from the construction that α_{-n} is homogeneous of bidegree $(n, |\alpha|)$.

For positive indices α_n can be defined by reading the same diagram backwards. I prefer to define α_n as the operator adjoint to α_{-n} with respect to the pairings on $\mathbb{H}^\ell$ given by $\langle x, y\rangle = (-1)^\ell \int_{X^{[\ell]}} x \cup y$. Note that this convention tallies with our definition for the pairing on the cohomology of $X = X^{[1]}$. Finally α_0 is set zero. Then Nakajima's main result is

Theorem 4.4. *(Nakajima* [**32**]*)— For all $\alpha, \beta \in H$ and $n, m \in \mathbb{Z}$,*

$$[\alpha_n, \beta_m] = n\delta_{n,m}\langle\alpha, \beta\rangle \mathrm{id}_{\mathbb{H}}. \tag{58}$$

We give a proof of this theorem in the next section.

We can rephrase the theorem by saying that the Heisenberg Lie algebra $\mathfrak{h}$ constructed in the previous section acts on $\mathbb{H}$ via the geometrically defined operators α_n. As this action is also faithful our otherwise irresponsible double usage of the symbol α_n as an element in $\mathfrak{h}$ and as an endomorphism on $\mathbb{H}$ is justified.

Proposition 4.5. — $\mathbb{V} \cong \mathbb{H}$ *as representations of* $\mathfrak{h}$.

Proof. The zeroeth Hilbert scheme is a point representing the empty subscheme of X. Its cohomology is generated by the unique element $\mathbf{1} := 1 \in H^0(X^{[0]})$. It is clear for dimension reasons that $\alpha_n \mathbf{1} = 0$ for all $n \geq 0$. Sending the generator $\mathbf{1} \in \mathbb{V}$ to $\mathbf{1} \in \mathbb{H}$ defines a homomorphism $f : \mathbb{V} \to \mathbb{H}$. Since $\mathbb{V}$ is irreducible f must be injective. It follows that the dimensions of the homogeneous components of V give lower bounds for the Betti numbers of the Hilbert schemes. But we have already seen that by Göttsches formula these numbers are equal. It follows that f is an isomorphism. $\square$

As I learned from Marc de Cataldo in this conference, the use of Göttsches formula can be avoided altogether by the following argument. Consider the Leray spectral sequence for the constant sheaf $\mathbb{Q}$ and the Hilbert-Chow morphism $\rho : \mathrm{Hilb}^n(X) \to S^n X$:

$$E_2^{pq} \implies H^{p+q}(\mathrm{Hilb}^n(X)), \tag{59}$$

where the E_2 term is given by $\bigoplus_\alpha \bigotimes_{i \in \mathbb{N}} S^{\alpha_i}(H)$, and α runs over all partitions $\alpha = (1^{\alpha_1} 2^{\alpha_2} \ldots)$ of n, i.e. $n = \sum_i \alpha_i i$. Göttsches formula is equivalent to the degeneracy of this spectral sequence. However, and this is de Cataldo's point: we need not know that the spectral sequence degenerates. The Betti numbers of the Hilbert schemes can only become *smaller* than the Betti numbers of E_2. On the other hand, Nakajima's theorem shows that the same numbers provide lower bounds. Thus we must have degeneracy of the sequence and $\mathbb{V} \cong \mathbb{H}$.

One important consequence of the isomorphism $\mathbb{V} \cong \mathbb{H}$ is that we can write down a natural basis for the cohomology of the Hilbert schemes that also has a geometrical interpretation. Let α^i be a basis of homogeneous element in H. Then $\mathbb{H}$ has a basis of elements

$$\alpha^{i_1}_{-n_1} \dots \alpha^{i_s}_{-n_s}\mathbf{1}, \tag{60}$$

where a factor $\alpha^{i_j}_{-n_j}$ should not show up twice if the degree of α^{i_j} is odd.

4.4. Proof of the Nakajima relations. We will sketch a proof for the commutator relations (58). The most interesting case is of course the bracket for operators α_n and β_{-n}. The cases where $n+m \neq 0$ are treated essentially in the same way, except that they are simpler: one never needs seriously compute an intersection product, dimension arguments suffice.
We want to calculate the two operators $\alpha_n\beta_{-n}$ and $\beta_{-n}\alpha_n$ on $\mathbb{H}$, or rather their difference.
Geometrically speaking, the operator $\alpha_n\beta_{-n}$ first adds an n-fold point somewhere and afterwards subtracts an n-fold point, perhaps somewhere else. In contrast, the operator $\beta_{-n}\alpha_n$ first subtracts an n-fold point, and then adds an n-fold point, possibly a different one and possibly somewhere else. Heuristically, it is evident that the second procedure is more difficult to realise geometrically: in order to subtract something of length n from a subscheme $\xi \subset X$, the local ring of ξ should first of all have length $\geq n$ at some point x of its support. This is not so in the first case: we can always at least subtract what we added in first.
Consider the product variety

$$Y_+ := X^{[\ell]} \times X \times X^{[\ell+n]} \times X \times X^{[\ell]} \tag{61}$$

and let p_{123} etc. denote the projection onto the product of the first three factors etc. The cycle

$$w_+ := p_{1245*}(p_{123}^*[Z^{\ell,\ell+n}].p_{345}^*[Z^{\ell,\ell+n}]) \in A_{2\ell+2}(X^{[\ell]} \times X \times X \times X^{[\ell]}) \tag{62}$$

defines the operator $\alpha_n\beta_{-n}$:

$$\alpha_n\beta_{-n}(y) = (-1)^n PD\big(p_{1*}(p_2^*(\alpha) \cup p_3^*(\beta) \cup p_4^*(y) \cap w_+\big). \tag{63}$$

Geometrically, w_+ is supported on the scheme

$$W_+ := p_{1245}(p_{123}^{-1}(Z^{\ell,\ell+n}) \cap p_{345}^{-1}(Z^{\ell,\ell+n})). \tag{64}$$

Set-theoretically, W_+ is easy to describe:

$$W_+ := \left\{(\xi, x, y, \zeta) \,\middle|\, \exists\, \eta \;:\; \xi \subset \eta \supset \zeta, \rho(\eta) = \rho(\xi) + nx = \rho(\zeta) + ny\right\}. \tag{65}$$

We can immediately identify two irreducible components of W_+ of dimension $2\ell+2$: the first is the closure W' of the image of the rational map

$$X^{[\ell-n]} \times B^n \times B^n \dashrightarrow X^{[\ell]} \times X \times X \times X^{[\ell]} \tag{66}$$

given by $(\eta, \eta', \eta'') \mapsto (\eta \cup \eta', \rho(\eta'), \rho(\eta''), \eta \cup \eta'')$ and defined on all pairwise disjoint triples (η, η', η''). The second is the diagonal

$$\Delta = \{(\xi, x, x, \xi) \mid \xi \in X^{[\ell]}, x \in X\} \subset X^{[\ell]} \times X \times X \times X^{[\ell]}. \tag{67}$$

A dimension argument essentially based on Briançon's theorem shows that any other component of W_+, if there is any at all, must be of dimension strictly less than $2\ell + 2$ and therefore cannot support a cycle class of this degree.

The cycle w_+ is used to compute the composite operator $\alpha_n\beta_{-n}$. For the composition in the opposite order we proceed analogously.

Consider the product variety

$$Y_- := X^\ell \times X \times X^{[\ell-n]} \times X \times X^{[\ell]} \tag{68}$$

and the cycle

$$w_- := p_{1245*}(p_{123}^*[Z^{\ell,\ell-n}].p_{345}^*[Z^{\ell,\ell-n}]) \;\in\; A_{2\ell+2}(X^{[\ell]} \times X \times X \times X^{[\ell]}). \tag{69}$$

Geometrically, w_- is supported by the variety

$$W_- := p_{1245}(p_{123}^{-1}(Z^{\ell,\ell-n}) \cap p_{345}^{-1}(Z^{\ell,\ell-n}). \tag{70}$$

The latter has the following set theoretic description:

$$W_- := \left\{(\xi, x, y, \zeta) \,\middle|\, \exists\, \theta \;:\; \xi \supset \theta \subset \zeta, \rho(\xi) = \rho(\theta) + nx, \rho(\zeta) = \rho(\theta) + ny\right\}. \tag{71}$$

As predicted by our heuristic statement at the very beginning, there is only one component of dimension $2\ell+2$, namely the variety W' defined above, and any other component of W_-, if there is any, must be of dimension strictly less than $2\ell + 2$.

Thus we must have $w_+ = a_+[W'] + N[\Delta]$ and $w_- = a_-[W]$ for some integers a_-, a_+, N. To compute these coefficients, it suffices to localise the geometric setting at a generic point of W' and Δ.

A generic point of W' has the form $(\eta \cup \eta', x, y, \eta \cup \eta'')$ where η, η' and η'' are disjoint, and η' and η'' have length n and are supported at disjoint point x and y, respectively. It is not difficult to check that the intersection of $p_{123}^{-1}Z^{\ell,\ell+n}$ and $p_{345}^{-1}Z^{\ell,\ell+n}$ in Y_+ is transverse above $(\eta \cup \eta', x, y, \eta \cup \eta'')$ and yields $a_+ = 1$, and that $a_- = 1$ for similar reasons.

This shows that $w_+ - w_- = N[\Delta]$. Note that the structure of this equation already proves (58) except for the precise value of the coefficient that appears on the right hand side in (58). This factor was first computed by Ellingsrud and Strømme [**8**] and then by a quite different and very interesting argument by Nakajima [**33**].
It remains to compute N: again we localise the geometric setting at a generic point of the diagonal, i.e. a point of the form (ξ, x, x, ξ) where x is disjoint from ξ. Consider a point (ξ, x, η, x, ξ) in Y_+ above (ξ, x, x, ξ). Up to an etale map, we may replace $X^{[\ell+n]}$ near (ξ, x, η, x, ξ) by $X^{[\ell]} \times X^{[n]}$, and $Z^{\ell,n+\ell}$ by $X^{[\ell]} \times B^n$ etc. In this way the whole situation becomes a product of $X^{[\ell]}$ with a much simpler scenario corresponding to the case $\ell = 0$. In other words, N does not depend on ℓ, and we may assume without loss of generality that $\ell = 0$.
The problem is reduced to the following: Let $i : B^n \to X^{[n]}$ denote the natural inclusion and $\rho : B^n \to X$ the map that sends ξ to its support. Moreover, let $j = (i, \rho) : B^{[n]} \to X^{[n]} \times X$ be the embedding as graph of ρ. We need to compute the intersection of $j(B^n) \times X$ and $X \times j(B^n)$ in $X \times X^{[n]} \times X$, or more precisely, the coefficient N in

$$p_{13*}\Big(j_*[j(B^n] \times [X].[X] \times j_*[B^n]\Big) = N[\Delta], \tag{72}$$

where $[\Delta]$ is the class of the diagonal in $X \times X$ and p_{13} is the projection from $X \times X^{[n]} \times X$ to $X \times X$. Intersect with $pr_3^{-1}(p)$ for some point $p \in X$ and get

$$p_{1*}\big(j_*[B^n].[X] \times [B^n_p]\big) = N[p], \tag{73}$$

where p_1 is the projection from $X \times X^{[n]}$ to X. This shows that N is the intersection number of $[B^n]$ and $[B^n_p]$ in $X^{[n]}$.

THEOREM 4.6. *(Ellingsrud-Strømme* [**8**]*)* —

$$[B^n].[B^n_p] = (-1)^{n-1} n. \tag{74}$$

PROOF. Recall the maps $\mathrm{Hilb}^n(X) \xleftarrow{\varphi} \mathbb{P} \xrightarrow{\psi} \mathrm{Hilb}^{n+1}(X)$. As before, let $E \subset \mathbb{P}$ denote the exceptional divisor of the blow-up $\Phi = (\phi, \rho) : \mathbb{P} \to \mathrm{Hilb}^n(X) \times X$. Moreover, let $B' := \psi^{-1}(B^{n+1})$ and $B'_p = \psi^{-1}(B^{n+1}_p)$ for some closed point $p \in X$. Then one has

$$\psi^*[B^{n+1}] = (n+1)\,[B'] \quad , \quad \psi^*[B^{n+1}_p] = (n+1)\,[B'_p] \tag{75}$$

$$[E].\phi^*[B^n] = n\,[B'] \quad , \quad [E].\phi^*[B^n_p] = n\,[B'_p] \tag{76}$$

The first of these identities follows easily from the fact that $\psi : \mathbb{P} \to \mathrm{Hilb}^{n+1}$ is a generically finite map of degree $n+1$. Moreover, it is clear that $E \cap \phi^{-1}(B^n) = B'$. Hence one needs to determine the factor λ in $[E].\phi^*[B^n] = \lambda\,[B']$. This can be

done by an explicit calculation in local coordinates for $\mathrm{Hilb}^n(\mathbb{C}^2)$ near the point $\xi = V((y, x^n))$ in B^n.
We can now prove the theorem by induction on n and assume that it has been shown for n.

$$\frac{1}{n+1}[B^{n+1}].[B_p^{n+1}] = \frac{1}{(n+1)^2}\psi^*[B^{n+1}].\psi^*[B_p^{n+1}] \tag{77}$$

$$= [B'].[B'_p] = \frac{1}{n^2}[E].\phi^*[B^n].[E].\phi^*[B_p^n] \tag{78}$$

$$= \frac{(-1)^n n}{n^2}[E]^2.[F], \tag{79}$$

where in the last line we have used the induction hypothesis and $[F]$ denotes a generic fibre of $\phi : \mathbb{P} \to \mathrm{Hilb}^n$. Such a generic fibre equals the surface X blown up in n distinct points. Therefore $[E]^2.[F] = -n$. It follows that $[B^{n+1}].[B_p^{n+1}] = (-1)^{n+1}(n+1)$. □

5. Vertex algebras

The irreducible $\mathfrak{h}$-representation $\mathbb{V}$ that we constructed in section 4.2 carries naturally the structure of a vertex algebra. In fact, it is one of easiest nontrivial vertex algebras around. By Nakajima's isomorphism $\mathbb{V} \to \mathbb{H}$, this vertex algebra structure is transplanted onto the cohomology of the Hilbert schemes. The impotant point is of course, that this is not just an empty formal procedure, but that certain features of the vertex algebra are reflected in the geometry, and conversely.
In the little time and space available to me in these lectures I cannot give more than a very superficial introduction to vertex algebras. I recommend the text books of Kac [**20**] and Frenkel and Ben-Zvi [**11**]. However, I would like to introduce just as much of some basic notions as to give some idea why the language of fields might be a good way of organising the algebraic structures that have appeared in the previous section and will be used in the next.
Vertex algebras are (usually infinite dimensional) vector spaces with a distinguished element $\mathbf{1}$ (the so-called vacuum element) and a countable number of products $-_{(n)}- : V \otimes V \to V$ referred to as the n-th product, $n \in \mathbb{Z}$. None of these products is commutative or associative, and the failure is measured in terms of all the other products. A good way to organise these data is in terms of endomorphism valued power series: For a fixed $a \in V$, there is an endomorphism $a_{(n)}- : V \to V$ for each $n \in \mathbb{Z}$, and we put all of them into formal sum

$$a(z) := \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathrm{End}(V)[[z, z^{-1}]]. \tag{80}$$

The unusual indexing is introduced so as to give $a_{(n)} = \mathrm{res}_z\, z^n a(z)$. In this way, we get a map, the so-called state-field correspondence,

$$Y : V \to \mathrm{End}(V)[[z, z^{-1}]], \quad a \mapsto a(z). \tag{81}$$

The first vertex algebra axiom on Y requires that for any a and b in V the formal V-valued power series $a(z)b = \sum_{n\in\mathbb{Z}} a_{(n)} b z^{-n-1}$ should have a finite pole order, i.e. we require that $a_{(n)}b = 0$ for $n \gg 0$. In this case, the power series $a(z)$ is called a field. The set of all fields is a linear subspace $\mathrm{QEnd}(V) \subset \mathrm{End}(V)[[z, z^{-1}]]$.
Here is an example: let $\mathbb{V}$ denote the representation of $\mathfrak{h}$ that we discussed in the previous sections. For each $a \in H$, the Lie algebra elements a_n, $n \in \mathbb{Z}$, act on V, and may be organised into an $\mathrm{End}(V)$ valued formal power series. The grading of V and homogeneity of the a_n imply that $\sum_{n\in\mathbb{Z}} a_n z^{-n-1} =: a(z)$ is a field for each $a \in H \subset V$.
We must resist the temptation to multiply two fields

$$a(z)b(z) \quad \stackrel{?}{=} \quad \sum_{p\in\mathbb{Z}} \left(\sum_{m+n+1=p} a_{(m)} b_{(n)} \right) z^{-p-1}, \tag{82}$$

for the inner sum is infinite, and the expression does not make sense. However, because of the field property of $a(z)$ and $b(z)$ the term $a_{(p-n-1)} b_{(n)} c$ vanishes for any given c and $n \gg 0$. This is not so when n becomes very negative. In this case $a_{(p-n-1)}$ becomes zero when evaluated on a fixed vector, except that in the present situation its argument changes as well, and we are lost. In quantum field theory there is a radical cure for this: Simply let the a_n's act first. The normal ordered product is defined as follows

$$:a(z)b(z): \quad := \quad a(z)_- b(z) - (-1)^{|a|\cdot|b|} b(z) a(z)_+, \tag{83}$$

where $a(z)_+ = \sum_{n\geq 0} a_{(n)} z^{-n-1}$ and $a(z)_- := \sum_{n<0} a_{(n)} z^{-n-1}$ are the principal and the holomorphic part of $a(z)$, respectively. It is not difficult to see that this definition makes sense and that the normal ordered product of two fields is again a field. It is less obvious that to make such a definition really is a reasonable thing to do. More generally, we define an n-th product of two fields $a, b \in \mathrm{QEnd}(V)$ for any $n \in \mathbb{Z}$ by

$$a(z)_{(n)} b(z) := \mathrm{res}_w \Big(w^n (1 - \tfrac{z}{w})^n a(w) b(z) - (-1)^{|a|\cdot|b|} (-z)^n (1 - \tfrac{w}{z})^n b(z) a(w) \Big), \tag{84}$$

where $(1 - \frac{w}{z})^n = 1 - n\frac{w}{z} + \binom{n}{2}(\frac{w}{z})^2 + \ldots$ etc. Again this is well-defined, and $a(z)_{(n)} b(z)$ is again a field. One can check that the (-1)-product of two fields is the normal ordered product. If $(V, Y, \mathbf{1})$ is a vertex algebra then $(a_{(n)}b)(z) = a(z)_{(n)} b(z)$.

Let us go back to the $\mathfrak{h}$-representation on $\mathbb{V}$. Choose a basis a^i of H and a dual basis b^i with respect to the symmetric bilinear form $\langle -,-\rangle$. To each element $a \in H$ we have associated a field $a(z)$. We may therefore consider the field

$$L(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2} := \quad \frac{1}{2}\sum_i : a^i(z)b^i(z) : . \tag{85}$$

If one is not accustomed to this kind of calculation it might take a while to verify that the coefficients L_n of the field $L(z)$ satisfy the following commutator relations

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}\,\mathrm{sdim}(H)\mathrm{id}_{\mathbb{V}}, \tag{86}$$

where $\mathrm{sdim}(H) := \dim H^{\mathrm{even}} - \dim H^{\mathrm{odd}}$ is the super dimension of H. The commutator relations (86) are characteristic of for the Virasoro algebra.

The Virasoro algebra is defined as follows: $\mathrm{Witt} := \mathrm{Der}(\mathbb{Q}[t,t^{-1}])$ is the Lie algebra of vector fields on a punctured disc. It has a basis $\ell_n := t^{-n-1}\frac{d}{dt}$ that satisfies the Lie bracket relations $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$. Apparently, Witt is a perfect Lie algebra and therefore has a universal central extension

$$0 \longrightarrow Z \longrightarrow \mathrm{Vir} \longrightarrow \mathrm{Witt} \longrightarrow 0. \tag{87}$$

It turns out that the centre of Vir is one-dimensional, say spanned by an element c, and that one may choose lifts $L_n \in \mathrm{Vir}$ of $\ell_n \in \mathrm{Witt}$ in such a way that the Lie brackets for the L_n have the normalised form

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}c. \tag{88}$$

Comparing this with (86), we may express the meaning of (86) by saying that there is a natural representation of Vir on $\mathbb{V}$ where the element c acts as a scalar. This scalar is often referred to as the central charge of the representation. In the present case it is $\mathrm{sdim}(H)$.

Assume now that H has in addition to its grading and its symmetric pairing a ring structure $\cup : H \otimes H \to H$ with unit 1 and a trace map $t : H \to \mathbb{C}$ such that $\langle \alpha, \beta\rangle = t(\alpha \cup \beta)$. Let $\Delta : H \to H \otimes H$ be the map adjoint to the cup product with respect to the pairing $\langle -,-\rangle$. For instance, $\Delta(1) = \sum a^i \otimes b^i$, where (α^i) and (β^i) are dual bases as above. The element $e := \sum_i \alpha^i \cup \beta^i \in H$ is called the Euler class of H. It is the image of 1 under the composite map $H \xrightarrow{Delta} H \otimes H \xrightarrow{\cup} H$. Furthermore we have $t(e) = \mathrm{sdim}(H)$.

The following construction generalises the definition of the field $L(z)$ above: For any $\gamma \in H$ write $\Delta(\gamma) = \sum \gamma' \otimes \gamma''$

$$L(\gamma)(z) := \sum_{n\in\mathbb{Z}} L_n(\gamma) z^{-n-2} := \quad \frac{1}{2}\sum_i : \gamma'(z)\gamma''(z) : . \tag{89}$$

This defines endomorphisms $L_n(\gamma) \in \mathrm{End}(\mathbb{V})$ that satisfy the twisted Virasoro relations

$$[L_n(\alpha), L_m(\beta)] = (n-m)L_{n+m}(\alpha \cup \beta) + \delta_{n,-m}\frac{n^3-n}{12}t(\alpha\cup\beta\cup e). \tag{90}$$

For $\alpha = \beta = 1$, one gets back the old relations (88). We will need these endomorphisms in the next section. Explicitly, one has

$$L_n(\gamma) = \frac{1}{2}\sum_i \sum_{\nu\in\mathbb{Z}} \gamma'_\nu \gamma''_{n-\nu} \tag{91}$$

for $n \neq 0$ and

$$L_0(\gamma) = \sum_i \sum_{\nu\in\mathbb{N}} \gamma'^i_\nu \gamma''_{-\nu}. \tag{92}$$

I leave it as an exercise to check that

$$[L_n(\alpha), \beta_m] = -m(\alpha\cup\beta)_{n+m}. \tag{93}$$

6. The ring structure

Nakajima's beautiful theorem gives us a little bit more than just an explicite isomorphism of vector spaces $\mathbb{V} \to \mathbb{H}$. It also allows one to effectively compute the cup product pairing on $\mathbb{H}^n$. Since α_n is adjoint to α_n, one has

$$\begin{aligned}
&\langle \alpha^{i_1}_{-n_1}\cdots\alpha^{i_s}_{-n_s}\mathbf{1}, \alpha^{j_1}_{-m_1}\cdots\alpha^{j_t}_{-m_t}\mathbf{1}\rangle \\
&\qquad = \langle \alpha^{i_2}_{-n_2}\cdots\alpha^{i_s}_{-n_s}\mathbf{1}, \alpha^{i_1}_{+n_1}\alpha^{j_1}_{-m_1}\cdots\alpha^{j_t}_{-m_t}\mathbf{1}\rangle \\
&\qquad = \sum_{\tau=1}^{t} n_1\delta_{n_1,m_\tau}\langle\alpha^{i_1},\alpha^{j_\tau}\rangle\langle \alpha^{i_2}_{-n_2}\cdots\alpha^{i_s}_{-n_s}\mathbf{1}, \alpha^{i_1}_{+n_1}\alpha^{j_1}_{-m_1}\cdots\hat{\alpha}^{j_\tau}_{n_\tau}\cdots\alpha^{j_t}_{-m_t}\mathbf{1}\rangle.
\end{aligned}$$

Inductively, we may reduce everything to $\langle \mathbf{1}, \mathbf{1}\rangle = 1$.

However, so far there is no relation between the representation of $\mathfrak{h}$ on $\mathbb{H}$ and the additional ring structure on $\mathbb{H}^n$, given by the cup product. To understand this ring structure is the goal of this section.

6.1. Multiplication operators.

The basic idea is to think of the cup product in terms of multiplication operators and to relate these with Nakajima's operators. Recall that $\mathrm{Hilb}^n(X)$ parameterises a universal family

$$\begin{array}{ccc}
\Xi_n \subset \mathrm{Hilb}^n(X)\times X & \xrightarrow{q} & X \\
{\scriptstyle p}\downarrow & & \\
\mathrm{Hilb}^n(X) & &
\end{array} \tag{94}$$

If F is a locally free sheaf on X, then $F^{[n]} := p_*(\mathcal{O}_{\Xi_n}\otimes q^*F)$ is locally free on $\mathrm{Hilb}^n(X)$ of rank $n\cdot\mathrm{rk}(F)$. Sheaves of this type are called tautological sheaves.

Applying this construction to the trivial sheaf weobtain the sheaf of algebras $\mathcal{A} = \mathcal{O}^{[n]}$ encountered before.

Following Li, Qin and Wang [**28**] one can generalise this construction from elements in K-theory to cohomology classes. For each $\alpha \in H$ let

$$\alpha^{[n]} := p_*(\mathrm{ch}(\mathcal{O}_{\Xi_n}) \cup q^*(\mathrm{td}(X) \cup \alpha)). \tag{95}$$

With this notation the Grothendieck-Riemann-Roch theorem says that

$$\mathrm{ch}(F^{[n]}) = (\mathrm{ch}(F))^{[n]}. \tag{96}$$

Define operators $m(\alpha) \in \mathrm{End}(\mathbb{H})$ by

$$m(\alpha)|_{\mathbb{H}^n}(y) = \alpha^{[n]} \cup y. \tag{97}$$

Note that even if α is homogeneous, $m(\alpha)$ is not so. As a consequence, the operator has weight 0 but is not homogeneous in the cohomological direction. Let the homogeneneous component of bidegree $(0, j)$ be denoted by $m(\alpha)_j$. For example, $m(1)_0|_{\mathbb{H}_n}$ is the multiplication by $\mathrm{rk}(\mathcal{A}) = n$, and $m(1)_2|_{\mathbb{H}^n}$ is the cup product with $c_1(\mathcal{A}) = -\frac{1}{2}[\partial \operatorname{Hilb}^n(X)]$. This operator is of particular importance and deserves its own symbol

$$\partial := m(1)_2 = c_1(\mathcal{O}^{[n]}). \tag{98}$$

Up to a factor $-\frac{1}{2}$, ∂ describes the intersection of a cohomology class with the divisor $\partial \operatorname{Hilb}^n(X)$.

Finally, let $c(F) \in \mathrm{End}(\mathbb{H})$ be the endomorphism with $c(F)|_{\mathbb{H}^n} = c(F^{[n]}) \cup -$ for a locally free sheaf F on X.

Every element in $\mathbb{H}$ is a linear combination of elements of the form

$$x = \alpha^{i_1}_{-n_1} \cdot \ldots \cdot \alpha^{i_s}_{-n_s} \mathbf{1}. \tag{99}$$

Then

$$m(\beta)(x) = \sum_{j=1^s} \alpha^{i_1}_{-n_1} \cdot \ldots [m(\beta), \alpha^{i_j}_{-n_j}] \ldots \cdot \alpha^{i_s}_{-n_s} \mathbf{1}. \tag{100}$$

$$+ \alpha^{i_1}_{-n_1} \cdot \ldots \cdot \alpha^{i_s}_{-n_s} m(\beta) \mathbf{1}. \tag{101}$$

The last summand is zero, since $\Xi_0 = \emptyset$ so that $m(\beta)\mathbf{1} = 0$. We can compute $m(\beta)(x)$ if we understand the commutators $[m(\beta), \alpha_{-n}]$.

Theorem 6.1. —

$$[m(\beta), \alpha_{-1}] = \sum_{\nu \geq 0} \frac{1}{\nu!} (\mathrm{ad}\, \partial)^\nu (\beta \cup \alpha)_{-1}. \tag{102}$$

For the special case $\beta = ch(F)$ this theorem can be found in [**24**], for arbitrary β it is due to Li, Qin and Wang [**28**]. Theorem 6.1 reduces the calculation of $[m(\beta), -]$ to the special case $[\partial, -]$. The proof of the theorem is in fact quite easy:

PROOF. Recall the incidence scheme $\mathbb{P}$ studied in section 3.3 with the maps $\psi : \mathbb{P} \to \mathrm{Hilb}^{n+1}(X)$, $\phi : \mathbb{P} \to \mathrm{Hilb}^n(X)$ and $\rho : \mathbb{P} \to X$. Passing from the sequence (25) of ideal sheaves to structure sheaves we obtain the global version of sequence (21):

$$0 \longrightarrow \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}}(1)|_Y \longrightarrow (\psi \times \mathrm{id}_X)^* \mathcal{O}_{\Xi_{n+1}} \longrightarrow (\phi \times \mathrm{id}_X)^* \mathcal{O}_{\Xi_n} \longrightarrow 0. \tag{103}$$

Applying $p_*(\mathrm{ch}(-) \cup \mathrm{pr}_X^*(\mathrm{td}(X) \cup \beta))$ to this sequence we obtain the identity

$$\psi^* \beta^{[n+1]} - \phi^* \beta^{[n]} = \mathrm{ch}(\mathcal{O}_{\mathbb{P}}(1)) \cup \rho^* \beta \tag{104}$$

in $H^*(\mathbb{P})$. Now, by definition, $\alpha_{-1}(y) = \psi_*(\rho^* \alpha \cup \phi^*(y))$. It follows that

$$\begin{aligned} [m(\beta), \alpha_{-1}](y) &= \beta^{[n+1]} \cup \psi_*(\rho^*\alpha \cup \phi^* y) - (-1)^{|\alpha|\,|\beta|} \psi_*(\rho^* \alpha \cup \phi^*(\beta^{[n]} \cup y)) \\ &= \psi_* \Big((\psi^* \beta^{[n+1]} - \phi^* \beta^{[n]}) \cup \rho^* \alpha \cup \phi^* y \Big) \\ &= \psi_* \Big(\mathrm{ch}\, \mathcal{O}_{\mathbb{P}}(1) \cup \rho^*(\beta \cup \alpha) \cup \phi^* y \Big), \end{aligned}$$

where we have used the identity (104) in the last line. From this one can already deduce what Li, Qin and Wang called the transfer property:

$$[m(\beta), \alpha_{-1}] \;=\; [m(1), (\beta \cup \alpha)_{-1}]. \tag{105}$$

Moreover, if in the calculation above we take $\beta = 1$ and the degree 2 part of $m(1)$, we get

$$[\partial, \alpha_{-1}](y) = \psi_*(c_1(\mathcal{O}_{\mathbb{P}}(1)) \cup \rho^* \alpha \cup \phi^* y), \tag{106}$$

and more generally,

$$(\mathrm{ad}\, \partial)^\nu (\alpha_{-1})(y) = \psi_*((c_1(\mathcal{O}_{\mathbb{P}}(1))^\nu \cup \rho^* \alpha \cup \phi^* y), \tag{107}$$

If we reinsert this expression into (105), it follows that

$$[m(\beta), \alpha_{-1}] \;=\; \sum_{\nu \geq 0} \frac{1}{\nu!} (\mathrm{ad}\, \partial)^\nu (\beta \cup \alpha)_{-1} \tag{108}$$

□

Let L be a line bundle on X. Let $c(L) \in \mathrm{End}(\mathbb{H})$ denote the operator, that is the multiplication with the total chern class $c(L^{[n]})$ of the tautological rank n bundle $L^{[n]}$ on $\mathbb{H}^n$. Note that $c(L)$ is invertible.

THEOREM 6.2. [**24**]—

$$c(L) \alpha_{-1} c(L)^{-1} = (c(L) \cup \alpha)_{-1} + [\partial, \alpha_{-1}]. \tag{109}$$

The proof is very similar to the proof of theorem 6.1 and will be omitted.

The following theorem is the key to the ring structure of the Hilbert scheme. The proof is too long to reproduce it here. In the theorem, $K \in H^0 = H^2(X)$ is the canonical class of the surface. Moreover, $L_n(\alpha)$ is the twisted Virasoro operator as defined by equation (89)

THEOREM 6.3. [**24**] — *Let* $n \in \mathbb{N}$. *Then*

$$[\partial, \alpha_{-n}] = -nL_n(\alpha) + \frac{n(n-1)}{2}(\alpha \cup K)_{-n}. \tag{110}$$

Observe that due to the relations (91) and (92), the right hand side in the theorem is a quadratic expression in Nakajima operators. In particular, iterated applications of the theorem will compute all expressions of the form $(\operatorname{ad}\partial)^\nu(\alpha_n)$.

Here are some immediate consequences of theorem 6.3:

COROLLARY 6.4. — 1. $[\partial, \alpha_{-1}] = -L_{-1}(\alpha)$.
2. *For all* $n \in \mathbb{N}$*:* $(\operatorname{ad}[\partial, 1_{-1}])^n(\alpha_{-1}) = (-1)^n\, n!\, \alpha_{-n-1}$.

PROOF. The first statement is only a special case of the theorem, and the second follows from iterated applications of the identity (93). □

This corollary has in turn important applications: since any Nakajima operator α_{-n} can be written as a linear combination of words in the symbols ∂ and β_{-1}, theorem 6.1 is, in principle, strong enough to compute any of the commutators $[m(\beta), \alpha_{-n}]$: firstly, one has to rewrite α_{-n} as such a linear combination. Secondly, using the fact that $m(\beta)$ must commute with ∂, since both are multiplication operators in the graded commutative ring $\mathbb{H}^n$, express the commutator with $m(\beta)$ in terms of commutators with ∂. Use theorem 6.3 to compute these. In practise, this is difficult to carry out. However, one can turn these instructions into a rigid algorithm that allows to compute any cup product in $\mathbb{H}^n$.

COROLLARY 6.5. — $\mathbb{H} = \partial(\mathbb{H}) + H_{-1}\mathbb{H}$.

COROLLARY 6.6. — *Let* L *be a line bundle on* X. *Then*

$$\sum_{n\geq 0} c(L^{[n]})q^n = \exp\left(\sum_{m>0} \frac{(-1)^{m-1}}{m} c(L)_{-m}q^m\right)\mathbf{1} \tag{111}$$

PROOF. The unit $1_{\mathrm{Hilb}^n(X)} \in \mathbb{H}^n$ equals $\frac{1}{n!}(1_X)_1^n\mathbf{1}$. Hence the left hand side in (111) equals

$$\sum_{n\geq 0} c(L^{[n]})q^n = c(L)\exp((1_X)_{-1}q)\mathbf{1} = \exp(c(L)(1_X)_1 c(L)^{-1}q)\mathbf{1}. \tag{112}$$

By theorem 6.2 and 6.3,

$$c(L)(1_X)_1 c(L)^{-1} = c(L)_{-1} + [\partial, (1_X)_{-1}] = c(L)_{-1} + L(1_X)_{-1} =: A. \tag{113}$$

We must show that

$$\exp(Aq)\mathbf{1} = \exp((\sum_{m>0} \frac{(-1)^{m-1}}{m} c(L)_{-m} q^m)\mathbf{1}. \tag{114}$$

It suffices to show that the right hand side satisfies the differential equation

$$\frac{\partial \mathfrak{X}}{\partial q} = A\mathfrak{X} \tag{115}$$

with initial condition $\mathfrak{X}(0) = \mathbf{1}$. I leave the verification to the reader. □

6.2. The affine plane. The affine plane is an ideal example for the study of the cohomology ring of the Hilbert scheme because it has trivial cohomology. Therefore, there are no complications arising from the topology of $\mathbb{C}^2$, and $H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$ becomes an essentially combinatorial object.

Remark 6.7. — However, there is one problem with $\mathbb{C}^2$: it is not projective. Therefore, we have to modify our presentation of Nakajima's work. I will only briefly indicate the necessary changes: if X is a smooth but not necessarily projective surface then the projection from the incidence variety Z in (56) to the factor $X^{[\ell+n]}$ is still proper. Therefore the operators α_{-n} are well-defined for cohomology classes $\alpha \in H^*(X)$. However, the projection to $X^{[\ell]}$ is not proper. In order to define α_n for $n \in \mathbb{N}$ one needs to take α from cohomology with compact support. In this way, one can build up a Heisenberg Lie algebra $\mathfrak{h} = t^{-1}H^*(X)[t^{-1}] \oplus tH^*_c(x)[t]$ that acts on $\mathbb{H}$. It remains true that $\mathbb{H}$ is isomorphic to $S^*(t^{-1}H^*(X)[t^{-1}])$.

Let $X = \mathbb{C}^2$. Then $H = H^*(\mathbb{C}^2)[2]$ is is one-dimensional. For better readability, we write $p_n := (1_X)_{-n}$. Thus

$$\mathbb{H} = \bigoplus_{n\geq 0} H^*(\mathrm{Hilb}^n(\mathbb{C}^2))[2n]) = \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

Here p_n has weight n and degree -2. For example,

$$H^*(\mathrm{Hilb}^4(\mathbb{C}^2))[8] = \mathbb{Q}\langle p_1^4, p_1^2 p_2, p_1 p_3, p_2^2, p_4\rangle,$$

and $\dim H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$ equals the number of partitions of n.

Let $\mathbb{Q}[S_n]$ denote the group ring of the symmetric group. Every permutation π admits a decomposition $\pi = z_1 \cdot \ldots \cdot z_s$ into disjoint cycles z_i of length λ_i with $\lambda_1 \geq \ldots \geq \lambda_s$. We refer to the partition $\lambda = (\lambda_1, \cdots, \lambda_s)$ of n as the cycle type of π.

The center $Z\mathbb{Q}[S_n]$ of $\mathbb{Q}[S_n]$ consists of all elements that are invariant under the conjugation action of S_n. It has a $\mathbb{Q}$-basis consisting of elements $e_\lambda := \sum_{\pi\vdash\lambda} \pi$, where the sum runs over all permutations π of cycle type λ.

Consider the map $\Phi : \mathbb{Q}[S_n] \to \mathbb{Q}[p_1, p_2, \ldots]$ which sends a permutation π of cycle type λ to the monomial

$$\Phi(\pi) = \frac{1}{n!} p_{\lambda_1} \cdot \ldots \cdot p_{\lambda_s}.$$

Then Φ induces an isomorphism of vector spaces

$$\Phi : Z\mathbb{Q}[S_n] \longrightarrow \mathbb{H}^n = \mathbb{Q}[p_1, p_2, \ldots]_n, \quad e_\lambda \mapsto \prod_{i \in \mathbb{N}} \frac{1}{m_i(\lambda)!} \left(\frac{p_i}{i}\right)^{m_i(\lambda)},$$

where $m_i(\lambda) = |\{j \mid \lambda_j = i\}|$.

Both $Z\mathbb{Q}[S_n]$ and $\mathbb{H}^n = H^*(\mathrm{Hilb}^n(\mathbb{C}^2))[2n]$ are rings. However, they cannot be isomorphic for trivial reasons: $Z\mathbb{Q}[S_n]$ has a $\mathbb{Q}$-basis of idempotent elements, whereas $\mathbb{H}^n$ has only one idempotent: $1_{\mathrm{Hilb}^n(\mathbb{C}^2)}$. Nevertheless, these two rings are curiously related:

Proposition 6.8. — $\Phi(\sum_{\pi \in S_n} \mathrm{sgn}(\pi)\pi) = c(\mathcal{O}^{[n]})$.

In other words, Φ sends the alternating character to the total chern class of the tautological sheaf $\mathcal{O}^{[n]} = \mathcal{A}$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Proof. It is a combinatorial exercise to see that

$$\Phi\left(\sum_{n \geq 0} \sum_{\pi \in S_n} \mathrm{sgn}(\pi)\pi q^n\right) = \exp\left(\sum_{m > 0} \frac{(-1)^{m-1}}{m} p_m q^m\right).$$

By corollary 6.6, the right hand side equals $\sum_{n \geq 0} c(\mathcal{O}^{[n]}) q^n$. □

The action of $\partial = c_1(\mathcal{O}^{[n]}) \cup -$ on $\mathbb{H} = \mathbb{C}[p_1, p_2, \ldots]$ can be described as a differential operator. The following proposition is a consequence of theorem 6.3:

Proposition 6.9. [**24**] —

$$\partial = \frac{-1}{2} \sum_{n, m > 0} nm\, p_{n+m} \frac{\partial}{\partial p_n} \frac{\partial}{\partial p_m}.$$

As Frenkel and Wang [**12**] pointed out, the differential operator on the right hand side is very similar to the following operator introduced by I. Goulden [**16**]:

$$G := \frac{1}{2} \sum_{n, m > 0} nm\, p_{n+m} \frac{\partial}{\partial p_n} \frac{\partial}{\partial p_m} + \frac{1}{2} \sum_{n, m > 0} (n+m)\, p_n p_m \frac{\partial}{\partial p_{n+m}}.$$

Let $\tau \in Z\mathbb{Q}[S_n]$ denote the sum of all transpositions. Then Goulden's operator describes the multiplication with τ in $Z\mathbb{Q}[S_n]$ as differential operator the same way as ∂ describes the multiplication with $c_1(\mathcal{O}^{[n]})$, namely:

$$\Phi(\tau \cdot y) = G(\Phi(y)), \quad c_1(\mathcal{O}^{[n]}) \cup \Phi(y) = \partial(\Phi(y)) \tag{116}$$

To see this observe the following: if a transposition (ij) is composed with a permutation $\pi = z_1 \cdot \ldots \cdot z_s$ two things can happen: either i and j belong to two different cycles z_k, then as a result of the multiplication the two cycles are merged. For instance

$$(14) \cdot (1\,2\,3)(4\,5) = (1\,2\,3\,4\,5). \tag{117}$$

Thus two cycles of length n and m are replaced by a cycle of length $n+m$. This is accounted for with the correct multiplicities by the first half of G. If on the other hand i and j belong to the same cycle, then this cycle will will be split, as in

$$(14) \cdot (1\,2\,3\,4\,5) = (1\,2\,3)(4\,5). \tag{118}$$

Thus one cycle of length $n+m$ is split into two cycles of length n and m. The second half of G takes care of this possibility.

As we already know that $\Phi(-\tau) = c_1(\mathcal{O}^{[n]})$, the similarity of the differential operators ∂ and G allows only the following conclusion: in order to obtain the correct cup product in $\mathbb{H}^n$, the multiplication in $Z\mathbb{Q}[S_n]$ has to be modified in such a way that only multiplications as in (117) remain nonzero. This can be achieved by introducing an appropriate grading.

Define the age of $\pi \in S_n$ as the minimal number k needed to write π as a product of k transpositions. Clearly, $\mathrm{age}(\pi) = n - s$, where s is the number of disjoint cycles in the cycle decomposition of π. Moreover, it if follows directly from the definition that $\mathrm{age}(\pi\sigma) \leq \mathrm{age}(\pi) + \mathrm{age}(\sigma)$. Let $F_i^{\mathrm{age}} Z\mathbb{Q}[S_n]$ be generated by all elements of age $\leq i$. Then $F_i \cdot F_j \subset F_{i+j}$. Hence, the subspaces F_i^{age} define an ascending filtration of $Z\mathbb{Q}[S_n]$, and we may pass to the associated graded ring $\mathrm{gr}^{\mathrm{age}}\, Z\mathbb{Q}[S_n]$.

The discussion above makes the following theorem look very plausible. The same result was obtained independently and by a quite different method by Vasserot [**37**]:

Theorem 6.10. — *The map*

$$\Phi : \mathrm{gr}^{\mathrm{age}}\, Z\mathbb{Q}[S_n] \cong H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$$

is an isomorphism of rings that is degree preserving if the age is doubled.

For a proof within the given framework I refer to the joint paper with Sorger [**26**]. The key point is that due to Proposition 6.8 we have $\Phi(\sum_{\pi\in S_n, \mathrm{age}(\pi)=i}) = c_i(\mathcal{O}^{[n]})$ and that these elements generate the rings $\mathrm{gr}^{\mathrm{age}}\, Z\mathbb{Q}[S_n]$ and $H^*(\mathrm{Hilb}^n(\mathbb{C}^2))$, respectively.

6.3. Orbifold cohomology. In order to generalise theorem 6.10 for other smooth surfaces, it is best to use the notion of orbifold cohomology rings. Orbifold cohomology itself has a longer history, but apparently the ring structure was first introduced by Chen and Ruan [**4**],[**35**]. We will need here only the case of global group quotients. For this case we refer also to the paper of Fantechi and Göttsche [**9**].

Let X be a smooth projective variety on which a finite group G acts in such a way, that ω_X is locally trivial as a G-linearised sheaf. Equivalently, the stabiliser group G_x of every point $x \in X$ is supposed to lie in $\mathrm{SL}(T_x X)$.

Let $g \in G$ be an element of order n. The fixed point set X^g is a smooth subvariety of X. The isomorphism class of the tangent space $T_x X$ considered as a representation of the cyclic group $\langle g \rangle$ is locally constant on X^g. Let $\exp(2\pi i k_j)$, $j = 1, \dots, \dim(X)$, denote the eigenvalues of $g|_{T_x X}$. The k_j are normalised by the requirement $k_j \in \mathbb{Q} \cap [0, 1)$. Then the age of g is the locally constant function $\mathrm{age}(g) : X^g \to \mathbb{N}_0$ with $\mathrm{age}(g)(x) = \sum_j k_j$.

Remark 6.11. — In the previous subsection we have already defined the age of a permutation π. The connection with the present definition is this: The symmetric group acts on $(\mathbb{C}^2)^n$ by permuting the factors. Let π be a permutation with disjoint cycle decomposition $\pi = z_1 \cdot \ldots \cdot z_s$, and let $\ell(z_i) =: \ell_i$. The fixed point set of π is isomorphic to $(\mathbb{C}^2)^s$, and the eigenvalues of π are $\exp(2\pi i k/\ell_p)$ with $k = 0, \dots, \ell_p - 1$ and $p = 1, \dots, s$, each of them counted twice. The sum of all these k/ℓ_p gives the age of π in the new sense, namely

$$\mathrm{age}(\pi) = 2 \sum_{p=1}^{s} \sum_{k=0}^{\ell_p - 1} k/\ell_p = \sum_{p=1}^{s} (\ell_p - 1) = n - s.$$

This agrees with the previous definition.

Consider $(m, \mathrm{pr}_2) : G \times X \to X \times X$, where m is the action of G on X, and define the inertia subvariety $I \subset G \times X$ by means of the cartesian diagram

$$\begin{array}{ccc} I & \longrightarrow & G \times X \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

In other words, $I = \bigcup_{g \in G} \{g\} \times X^g$. We may consider age as a locally constant function on I. The group G acts naturally on $G \times X$ (by conjugation on the first factor) and I is a G-invariant subspace. Let

$$H^*(X, G) := H^*(I)[-2\,\mathrm{age}] = \bigoplus_{g \in G} H^*(X^g)[-2\,\mathrm{age}(g)]$$

and

$$H^*_{\mathrm{orb}}(X/G) := H^*(X, G)^G = \bigoplus_{[g] \in G/\sim} H^*(X^g)^{C_g}[-2\,\mathrm{age}(g)],$$

where in the last sum g runs through a set of representatives for the conjugacy classes of G, and $C_g \subset G$ is the centraliser of g.

This defines the orbifold cohomology of X/G as a vector space. It contains the ordinary cohomology $H^*(X/G)$ as the direct summand corresponding to the neutral element of G. The other direct summands are called twisted sectors in the physically inspired terminology.

In order to define a product on $H^*_{\mathrm{orb}}(X/G)$ one first defines a product on $H^*(X,G)$ and shows then that this product is G-equivariant. Define a map

$$m : H^*(X^g)[-2\,\mathrm{age}(g)] \times H^*(X^h)[-2\,\mathrm{age}(h)] \longrightarrow H^*(X^{gh})[-2\,\mathrm{age}(gh)]$$

by

$$m(\alpha,\beta) = i_*(\alpha|_{X^{g,h}} \cup \beta|_{X^{g,h}} \cup c_{g,h}),$$

where $X^{g,h} = X^g \cap X^h$ is the fixed point set of the subgroup $H \subset G$ generated by g and h, $i : X^{g,h} \to X^{gh}$ is the inclusion map, and $c_{g,h} \in H^*(X^{g,h})$ is a certain cohomology class still to be defined:

There is a unique ramified cover $C \to \mathbb{P}^1$ with Galois group H, ramified over $\{0,1,\infty\}$, such that the inertia groups over the points $0,1$ and ∞ are generated by g, h, and gh, respectively. Now

$$c_{g,h} := c_{\mathrm{top}}\Big((TX|_{X^{g,h}} \otimes_{\mathbb{C}} H^1(C,\mathcal{O}_C))^H\Big).$$

On the right hand, $TX|_{X^{g,h}}$ is an H-bundle that is twisted with the H-representation $H^1(C,\mathcal{O}_C)$, and $c_{g,h}$ is the top Chern class of the H-invariant part of this bundle. The following theorem is by no means obvious [**9**].

Theorem 6.12. *(Fantechi-Göttsche)— m preserves the grading and is associative.*

So finally, $H^*(X,G)$ has got a ring structure. It is not difficult to check that this ring structure is G-equivariant and therefore gives rise to a well-defined ring structure on the orbifold cohomology $H^*_{\mathrm{orb}}(X/G)$.

Theorem 6.13. — *Let X be a smooth projective surface with numerically trivial canonical divisor. Then there is a ring isomorphism*

$$H^*(\mathrm{Hilb}^n(X)) \cong H^*_{\mathrm{orb}}(X^n/S_n).$$

I will restrict myself to a few comments on the logical dependence of the various ingredients for the proof:

1. The theorem is not literally true unless one modifies the definition of the orbifold cohomology by making certain sign twists, see [**9**].
2. The isomorphism of the underlying graded vector spaces follows directly from Göttsche's formula.
3. Li, Qin and Wang [**28**],[**29**] have shown that the homogeneous components of the elements $\alpha^{[n]}$, $\alpha \in H^*(X)$, generate $H^*(\mathrm{Hilb}^n(X))$ as a ring.
4. Sorger and the author [**27**] constructed an abstract ring $H^*(X)^{[n]}$ and showed that $H^*(\mathrm{Hilb}^n(X)) \cong H^*(X))^{[n]}$. The key point being again the explicit formula of $c(\mathcal{O}_X^{[n]})$.

5. Fantechi and Göttsche [**9**], and independently, Uribe [**36**], computed the orbifold cohomology ring $H^*_{\mathrm{orb}}(X^n/S_n)$ and showed that $H^*_{\mathrm{orb}}(X^n/S_n) \cong H^*(X))^{[n]}$.

Ruan proposed the following more general conjecture, of which the theorem above then verifies a special case:

CONJECTURE 6.14. *(Ruan)— Let Y be a symplectic complex manifold and assume that the group G acts on Y preserving the symplectic structure. Assume furthermore that the quotient Y/G admits a crepant resolution $Z \to Y/G$. Then there is a ring isomorphism $H^*Z \cong H^*_{\mathrm{orb}}(Y/G)$.*

Besides the case of the Hilbert-Chow map $\rho : \mathrm{Hilb}^n(X) \to X^n/S_n$ for $K3$ and abelian surfaces that is covered by theorem 6.13 the following instances of Ruan's conjecture have been proved:

- The Kummer varieties $K_{n-1}(A)$ associated to an abelian surface A resolve the quotient A^{n-1}/S_n for the acion of S_n on the subvariety $\cong A^{n-1}$ of all points in A^n of sum 0. This case was done by Britze [**3**] in his Ph.D. thesis, using calculations of Fantechi and Göttsche [**9**].
- Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. The wreath product $\Gamma_n = \Gamma \wr S_n$ acts on $(\mathbb{C}^2)^n$, and the n-th Hilbert scheme of a minimal resolution of $\mathbb{C}^2/\Gamma$ provides a crepant resolution of the quotient $\mathbb{C}^{2n}/\Gamma_n = S^n(\mathbb{C}^2/\Gamma)$. The conjecture was proved for this case by Wang [**39**].
- More recently, Ginzburg and Kaledin [**21**] proved the conjecture for the linear case, i.e. a symplectic action of a finite group G on a symplectic vector space V. The methods are very interesting and quite different from those discussed above.

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